

The monoidal category of Yetter-Drinfeld modules over a weak braided Hopf algebra

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Abstract

In this paper we introduce the notion of weak operator and the theory of Yetter-Drinfeld modules over a weak braided Hopf algebra with invertible antipode in a strict monoidal category. We prove that the class of such objects constitutes a non strict monoidal category. It is also shown that this category is not trivial, that is to say that it admits objects generated by the adjoint action (coaction) associated to the weak braided Hopf algebra.

Keywords. Monoidal category, weak braided Hopf algebra, weak operator, Yetter-Drinfeld module.

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INTRODUCTION

The notion of Yetter-Drinfeld module was considered to deal with the quantum Yang-Baxter equation, specially in quantum mechanics (see [15] for a detailed exposition of its physical implications). Actually, every Yetter-Drinfeld module gives rise to a solution to the quantum Yang-Baxter equation, as was proved in [13], and if H is a finite Hopf algebra in a symmetric category \mathcal{C} , the category ${}^H_H\mathcal{YD}$ of left-left Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld quantum double, which was originally conceived to find solutions of the Yang-Baxter equation via universal matrices. Continuing with physical applications, any projection of a Hopf algebra provides an example of a Yetter-Drinfeld module (see [19]) and this result is the substrate of the bosonization process introduced by Majid in [14] that gives, for a quasitriangular Hopf algebra, an interpretation of cross products in terms of quantum algebras of observables of dynamical systems, as well as in quantum group gauge theory.

On the other hand, weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [17]) were introduced by Böhm, Nill and Szlachányi in [7] as a new generalization of Hopf algebras and groupoid algebras. The main difference with other Hopf algebraic

constructions, such as quasi-Hopf algebras and rational Hopf algebras, is that weak Hopf algebras are coassociative but the coproduct is not required to preserve the unit or, equivalently, the counit is not an algebra morphism. Some motivations to study weak Hopf algebras come from the following facts: firstly, as group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras; secondly, these algebraic structures have a remarkable connection with the theory of algebra extensions, important applications in the study of dynamical twists of Hopf algebras and a deep link with quantum field theories and operator algebras (see [17]), as well as they are useful tools in the study of fusion categories in characteristic zero (see [10]). The theory of Yetter-Drinfeld modules for a weak Hopf algebra was introduced by Böhm in [8]. Later, Nenciu proved in [16] that this category is isomorphic to the category of modules over the Drinfeld quantum double (the interested reader can also see [9]).

In [1] we can find the extension of Radford's theory for projections of Hopf algebras to projections of weak Hopf algebras in a strict symmetric monoidal category \mathcal{C} where every idempotent morphism splits. The main result of [1], extended to the braided setting in [5], assures that there exists a categorical equivalence between the category of isomorphism classes of projections associated to a weak Hopf algebra H and the category of Hopf algebras in the category of left-left Yetter-Drinfeld modules over H . To show this result, the authors introduced in [1] the notions of weak Yang-Baxter operator and weak braided Hopf algebra. Roughly speaking, a weak braided Hopf algebra in a strict monoidal category is an algebra-coalgebra with a weak Yang-Baxter operator, satisfying some compatibility conditions. This definition generalizes the one introduced by Takeuchi in [21], i.e., the definition of braided Hopf algebra, and the classical notions of Hopf algebra and Hopf algebra in a braided category. Moreover, as particular instances we recover the definition of weak Hopf algebra and, if the weak Yang-Baxter operator is the braiding of a braided category, the notion of weak Hopf algebra in a braided monoidal setting is formulated. The first non-trivial example of weak braided Hopf algebras can be constructed by modifying the algebraic structure of a Hopf algebra D in the non-strict braided monoidal category ${}^H_H\mathcal{YD}$ [1, Corollary 2.14]. In this case with these new product, coproduct, unit, counit and antipode D is not a Hopf algebra neither a weak Hopf algebra in the usual sense.

In [5] the authors proved that some relevant properties about projections associated to a weak braided Hopf algebra can be obtained without the use of a general braiding in the category where the weak braided Hopf algebra lives. This fact motivates the following questions: is it possible to establish a Yetter-Drinfeld module category for a weak braided Hopf algebra in a general strict monoidal category where every idempotent morphism splits? is it this category isomorphic to the center of some monoidal category of modules? The positive answer to the first question is the main contribution of this paper. To do it we introduce the notion of weak operator which constitutes a generalization of the concept of weak Yang-Baxter operator and is the key in order to define a non-strict monoidal category of Yetter-Drinfeld modules associated to a weak braided Hopf algebra. To illustrate this new notions we provide several examples of Yetter-Drinfeld modules in this general setting. A family of them comes from projections of weak braided Hopf algebras, while another collection is based on the use of the adjoint (co)action that in the weak setting is not in general a (co)module structure for the weak braided Hopf algebra.

The organization of the paper is the following. In Section 1 the general framework is stated recalling the definitions of weak Yang-Baxter operator, weak braided bialgebra and weak braided Hopf algebra; then we introduce the notion of weak operator and obtain its main properties. In Section 2 we establish the definition of left-left Yetter-Drinfeld module over an arbitrary weak

braided Hopf algebra D and prove that these objects constitute a non strict monoidal category, giving explicitly all the required constraints and the base object. Section 3 is devoted to the study of projections and the relation between weak Yang-Baxter operators and weak entwining structures in terms of weak operators. Finally, in Section 4 we use the adjoint (co)action to obtain different examples of Yetter-Drinfeld structures starting from an arbitrary weak braided Hopf algebra and include the explicit computations for the particular cases of groupoid algebras, Frobenius separable algebras in a braided setting and projections of weak braided Hopf algebras.

1. WEAK OPERATORS

In this paper we denote a monoidal category \mathcal{C} as $(\mathcal{C}, \otimes, K, \mathbf{a}, \mathbf{l}, \mathbf{r})$ where \mathcal{C} is a category and \otimes (tensor product) provides \mathcal{C} with a monoidal structure with unit object K whose associative constraint is denoted by \mathbf{a} and whose left and right unit constraints are given by \mathbf{l} and \mathbf{r} respectively.

We denote the class of objects of \mathcal{C} by $|\mathcal{C}|$ and for each object $M \in |\mathcal{C}|$, the identity morphism by $id_M : M \rightarrow M$. For simplicity of notation, given objects M, N, P in \mathcal{C} and a morphism $f : M \rightarrow N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

From now on we assume that \mathcal{C} is strict and every idempotent morphism in \mathcal{C} splits, i.e. for every morphism $\nabla_Y : Y \rightarrow Y$ such that $\nabla_Y = \nabla_Y \circ \nabla_Y$ there exist an object Z (called the image of ∇_Y) and morphisms $i_Y : Z \rightarrow Y$ and $p_Y : Y \rightarrow Z$ such that $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$. There is not loss of generality in assuming the strict character for \mathcal{C} because of it is well known that given a monoidal category we can construct a strict monoidal category \mathcal{C}^{st} which is tensor equivalent to \mathcal{C} (see [12] for the details); neither in assuming that \mathcal{C} admits split idempotents, having into account that for a given category \mathcal{C} there exists an universal embedding $\mathcal{C} \rightarrow \hat{\mathcal{C}}$ such that $\hat{\mathcal{C}}$ admits split idempotents, as was proved in [12].

A braided monoidal category \mathcal{C} means a monoidal category in which there is, for all M and N in \mathcal{C} , a natural isomorphism $c_{M,N} : M \otimes N \rightarrow N \otimes M$, called the braiding, satisfying the Hexagon Axiom (see [11] for generalities). If the braiding satisfies $c_{N,M} \circ c_{M,N} = id_{M \otimes N}$ for all M, N in \mathcal{C} , the category will be called symmetric.

Definition 1.1. An algebra in \mathcal{C} is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in \mathcal{C} and $\eta_A : K \rightarrow A$ (unit), $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in \mathcal{C} such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \rightarrow B$ is an algebra morphism if $f \circ \eta_A = \eta_B$, $\mu_B \circ (f \otimes f) = f \circ \mu_A$.

A coalgebra in \mathcal{C} is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in \mathcal{C} and $\varepsilon_D : D \rightarrow K$ (counit), $\delta_D : D \rightarrow D \otimes D$ (coproduct) are morphisms in \mathcal{C} such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \rightarrow E$ is a coalgebra morphism if $\varepsilon_E \circ f = \varepsilon_D$, $(f \otimes f) \circ \delta_D = \delta_E \circ f$.

If A is an algebra, B is a coalgebra and $\alpha : B \rightarrow A$, $\beta : B \rightarrow A$ are morphisms, we define the convolution product by $\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B$.

If (D, η_D, μ_D) is an algebra in \mathcal{C} , the pair (M, φ_M) , with $M \in |\mathcal{C}|$ and $\varphi_M : D \otimes M \rightarrow D$ is said to be a left D -module if $\varphi_M \circ (\eta_D \otimes M) = id_M$ and $\varphi_M \circ (D \otimes \varphi_M) = \varphi_M \circ (\mu_D \otimes M)$. Given two left D -modules (M, φ_M) and (N, φ_N) , $f : M \rightarrow N$ is a morphism of left D -modules if $\varphi_N \circ (D \otimes f) = f \circ \varphi_M$.

If $(D, \varepsilon_D, \delta_D)$ is a coalgebra in \mathcal{C} , the pair (M, ϱ_M) with $M \in |\mathcal{C}|$ and $\varrho_M : M \rightarrow D \otimes M$ is said to be a left D -comodule if $(\varepsilon_D \otimes M) \circ \varrho_M = id_M$ and $(D \otimes \varrho_M) \circ \varrho_M = (\delta_D \otimes M) \circ \varrho_M$. Given

two left D -comodules (M, ϱ_M) and (N, ϱ_N) , $f : M \rightarrow N$ is a morphism of left D -comodules if $\varrho_N \circ f = (D \otimes f) \circ \varrho_M$. The notions of right D -(co)module are defined analogously.

Definition 1.2. Let D be in \mathcal{C} and let $t_{D,D} : D \otimes D \rightarrow D \otimes D$ be a morphism in \mathcal{C} . We will say that $t_{D,D}$ satisfies the Yang-Baxter equation if

$$(t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}). \quad (1)$$

Weak Yang-Baxter operators are generalizations of Yang-Baxter operators (see [11]) and were introduced by Alonso, González and Rodríguez in [1]. In [5] we prove that one axiom of the original definition can be dropped. We rewrite the improved definition:

Definition 1.3. Let D be in \mathcal{C} . A weak Yang-Baxter operator is a morphism $t_{D,D} : D \otimes D \rightarrow D \otimes D$ in \mathcal{C} such that:

- (a1) $t_{D,D}$ satisfies the Yang-Baxter equation.
- (a2) There exists an idempotent morphism $\nabla_{D,D} : D \otimes D \rightarrow D \otimes D$ satisfying the following identities:
 - (a2-1) $(\nabla_{D,D} \otimes D) \circ (D \otimes \nabla_{D,D}) = (D \otimes \nabla_{D,D}) \circ (\nabla_{D,D} \otimes D)$,
 - (a2-2) $(\nabla_{D,D} \otimes D) \circ (D \otimes t_{D,D}) = (D \otimes t_{D,D}) \circ (\nabla_{D,D} \otimes D)$,
 - (a2-3) $(t_{D,D} \otimes D) \circ (D \otimes \nabla_{D,D}) = (D \otimes \nabla_{D,D}) \circ (t_{D,D} \otimes D)$,
 - (a2-4) $t_{D,D} \circ \nabla_{D,D} = \nabla_{D,D} \circ t_{D,D} = t_{D,D}$.
- (a3) There exists a morphism $t'_{D,D} : D \otimes D \rightarrow D \otimes D$ such that:
 - (a3-1) The morphism $p_{D,D} \circ t_{D,D} \circ i_{D,D} : D \times D \rightarrow D \times D$ is an isomorphism with inverse $p_{D,D} \circ t'_{D,D} \circ i_{D,D} : D \times D \rightarrow D \times D$, where $p_{D,D}$ and $i_{D,D}$ are the morphisms such that $i_{D,D} \circ p_{D,D} = \nabla_{D,D}$ and $p_{D,D} \circ i_{D,D} = id_{D \times D}$ being $D \times D$ the image of $\nabla_{D,D}$.
 - (a3-2) $t'_{D,D} \circ \nabla_{D,D} = \nabla_{D,D} \circ t'_{D,D} = t'_{D,D}$.

Note that if $\nabla_{D,D} = id_{D \otimes D}$ then $t_{D,D}$ is an isomorphism and we recover the definition of Yang-Baxter operator introduced by Joyal and Street in [11]. Also, by [[1], Definition 1.2 and Proposition 1.3], we get that $t_{D,D}$ is a weak Yang-Baxter operator with associated idempotent $\nabla_{D,D}$ if and only if so is $t'_{D,D}$. Moreover, in this case

$$t'_{D,D} \circ t_{D,D} = t_{D,D} \circ t'_{D,D} = \nabla_{D,D}. \quad (2)$$

Finally, using the identities (2)-(5) of [1] we obtain:

$$(D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes t'_{D,D}) = (t'_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D), \quad (3)$$

$$(t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (t'_{D,D} \otimes D) = (D \otimes t'_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}), \quad (4)$$

$$(D \otimes t'_{D,D}) \circ (t'_{D,D} \otimes D) \circ (D \otimes t_{D,D}) = (t_{D,D} \otimes D) \circ (D \otimes t'_{D,D}) \circ (t'_{D,D} \otimes D), \quad (5)$$

$$(t'_{D,D} \otimes D) \circ (D \otimes t'_{D,D}) \circ (t_{D,D} \otimes D) = (D \otimes t_{D,D}) \circ (t'_{D,D} \otimes D) \circ (D \otimes t'_{D,D}). \quad (6)$$

Examples 1.4. (1) In this first example we assume that \mathcal{C} is symmetric. The categories of Yetter-Drinfeld modules over weak Hopf algebras provide non-trivial examples of weak Yang-Baxter operators. A weak Hopf algebra H is an object in \mathcal{C} with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

- (i) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$,
- (ii) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
 $= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \otimes \delta_H) \otimes H)$,
- (iii) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
 $= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(iv) There exists a morphism $\lambda_H : H \rightarrow H$ in \mathcal{C} (called the antipode of H) verifying:

$$(iv-1) \quad id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$(iv-2) \quad \lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$(iv-3) \quad \lambda_H \wedge id_H \wedge \lambda_H = \lambda_H.$$

If we define the morphisms Π_H^L (target), Π_H^R (source), as

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

it is straightforward to show that they are idempotent.

The first family of examples of weak Hopf algebras comes from the theory of groupoid algebras. Recall that a groupoid G is simply a small category where all morphisms are isomorphisms. In this example, we consider finite groupoids, i.e. groupoids with a finite number of objects. The set of objects of G , called also the base of G , will be denoted by G_0 and the set of morphisms by G_1 . The identity morphism on $x \in G_0$ will be denoted by id_x and for a morphism $\sigma : x \rightarrow y$ in G_1 , we write $s(\sigma)$ and $t(\sigma)$, respectively for the source and the target of σ .

Let G be a groupoid and R a commutative ring. The groupoid algebra is the direct product in $R\text{-Mod}$

$$RG = \bigoplus_{\sigma \in G_1} R\sigma$$

where the product of two morphisms is equal to their composition if the latter is defined and 0 otherwise, i.e. $\mu_{RG}(\tau \otimes \sigma) = \tau \circ \sigma$ if $s(\tau) = t(\sigma)$ and $\mu_{RG}(\tau \otimes \sigma) = 0$ if $s(\tau) \neq t(\sigma)$. The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. The algebra RG is a cocommutative weak Hopf algebra, with coproduct δ_{RG} , counit ε_{RG} and antipode λ_{RG} given by the formulas:

$$\delta_{RG}(\sigma) = \sigma \otimes \sigma, \quad \varepsilon_{RG}(\sigma) = 1, \quad \lambda_{RG}(\sigma) = \sigma^{-1}.$$

For the weak Hopf algebra RG target and source morphisms are respectively,

$$\Pi_{RG}^L(\sigma) = id_{t(\sigma)}, \quad \Pi_{RG}^R(\sigma) = id_{s(\sigma)}$$

and $\lambda_{RG} \circ \lambda_{RG} = id_{RG}$.

If (M, φ_M) and (N, φ_N) are left H -modules we denote by $\varphi_{M \otimes N}$ the morphism $\varphi_{M \otimes N} : H \otimes M \otimes N \rightarrow M \otimes N$ defined by

$$\varphi_{M \otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

For two left H -comodules (M, ϱ_M) and (N, ϱ_N) , we denote by $\varrho_{M \otimes N}$ the morphism $\varrho_{M \otimes N} : M \otimes N \rightarrow H \otimes M \otimes N$ defined by

$$\varrho_{M \otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\varrho_M \otimes \varrho_N).$$

Let (M, φ_M) , (N, φ_N) be left H -modules. The morphism

$$\nabla_{M \otimes N} = \varphi_{M \otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \rightarrow M \otimes N$$

is an idempotent. In this setting we denote by $M \times N$ the image of $\nabla_{M \otimes N}$ and by $p_{M \otimes N} : M \otimes N \rightarrow M \times N$, $i_{M \otimes N} : M \times N \rightarrow M \otimes N$ the morphisms such that $i_{M \otimes N} \circ p_{M \otimes N} = \nabla_{M \otimes N}$ and $p_{M \otimes N} \circ i_{M \otimes N} = id_{M \times N}$. It is not difficult to see that the object $M \times N$ is a left H -module with action $\varphi_{M \times N} = p_{M \otimes N} \circ \varphi_{M \otimes N} \circ (H \otimes i_{M \otimes N}) : H \otimes M \times N \rightarrow M \times N$.

In a similar way, if (M, ϱ_M) and (N, ϱ_N) are left H -comodules the morphism

$$\nabla'_{M \otimes N} = (\varepsilon_H \otimes M \otimes N) \circ \varrho_{M \otimes N} : M \otimes N \rightarrow M \otimes N$$

is an idempotent. We denote by $M \odot N$ the image of $\nabla'_{M \otimes N}$ and by $p'_{M \otimes N} : M \otimes N \rightarrow M \odot N$, $i'_{M \otimes N} : M \odot N \rightarrow M \otimes N$ the morphisms such that $i'_{M \otimes N} \circ p'_{M \otimes N} = \nabla'_{M \otimes N}$ and $p'_{M \otimes N} \circ i'_{M \otimes N} = id_{M \odot N}$. In a similar way to the preceding case, $M \odot N$ is a left H -comodule with coaction $\varrho_{M \odot N} = (H \otimes p'_{M \otimes N}) \circ \varrho_{M \otimes N} \circ i'_{M \otimes N} : M \odot N \rightarrow H \otimes (M \odot N)$.

We shall denote by ${}^H_H\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H , e.g.; (M, φ_M, ρ_M) is an object in ${}^H_H\mathcal{YD}$ if (M, φ_M) is a left H -module, (M, ρ_M) is a left H -comodule and

$$\begin{aligned} \text{(yd1)} \quad \rho_M &= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \rho_M) \circ (\eta_D \otimes M). \\ \text{(yd2)} \quad &(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\rho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M) \\ &= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \rho_M). \end{aligned}$$

Let M, N be in ${}^H_H\mathcal{YD}$. The morphism $f : M \rightarrow N$ is a morphism of left-left Yetter-Drinfeld modules if $f \circ \varphi_M = \varphi_N \circ (D \otimes f)$ and $(D \otimes f) \circ \rho_M = \rho_N \circ f$.

If $(M, \varphi_M, \varrho_M)$ is a left-left Yetter-Drinfeld module over H then it obeys the following equality [[1], Proposition 1.12]:

$$\nabla_{M \otimes N} = \nabla'_{M \otimes N}. \quad (7)$$

Then if the antipode of H is an isomorphism the category ${}^H_H\mathcal{YD}$ is a non-strict braided monoidal category. We expose briefly the braided monoidal structure.

For two left-left Yetter-Drinfeld modules $(M, \varphi_M, \varrho_M)$, $(N, \varphi_N, \varrho_N)$ the tensor product is defined as the image of $\nabla_{M \otimes N}$. By (7), $M \times N = M \odot N$ and this object is a left-left Yetter-Drinfeld module with the following action and coaction:

$$\begin{aligned} \varphi_{M \times N} &= p_{M \otimes N} \circ \varphi_{M \otimes N} \circ (H \otimes i_{M \otimes N}) \\ \varrho_{M \times N} &= (H \otimes p_{M \otimes N}) \circ \varrho_{M \otimes N} \circ i_{M \otimes N}. \end{aligned}$$

The base object is the image of the target morphism denoted by H_L , which is a left-left Yetter-Drinfeld module with (co)module structure

$$\varphi_{H_L} = p_L \circ \mu_H \circ (H \otimes i_L), \quad \varrho_{H_L} = (H \otimes p_L) \circ \delta_H \circ i_L,$$

where $p_L : H \rightarrow H_L$ and $i_L : H_L \rightarrow H$ are the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$.

The unit constraints are:

$$\mathfrak{l}_M = \varphi_M \circ (i_L \otimes M) \circ i_{H_L \otimes M} : H_L \times M \rightarrow M,$$

$$\mathfrak{r}_M = \varphi_M \circ c_{M,H} \circ (M \otimes (\overline{\Pi}_H^L \circ i_L)) \circ i_{M \otimes H_L} : M \times H_L \rightarrow M.$$

These morphisms are isomorphisms with inverses:

$$\mathfrak{l}_M^{-1} = p_{H_L \otimes M} \circ (p_L \otimes \varphi_M) \circ ((\delta_H \circ \eta_H) \otimes M) : M \rightarrow H_L \times M,$$

$$\mathfrak{r}_M^{-1} = p_{M \otimes H_L} \circ (\varphi_M \otimes p_L) \circ (H \otimes c_{H,M}) \circ ((\delta_H \circ \eta_H) \otimes M) : M \rightarrow M \times H_L.$$

If M, N, P are objects in the category ${}^H_H\mathcal{YD}$, the associativity constrain is defined by

$$\mathfrak{a}_{M,N,P} = p_{(M \times N) \otimes P} \circ (p_{M \otimes N} \otimes P) \circ (M \otimes i_{N \otimes P}) \circ i_{M \otimes (N \times P)} : M \times (N \times P) \rightarrow (M \times N) \times P.$$

and its inverse is

$$\mathfrak{a}_{M,N,P}^{-1} = p_{M \otimes (N \times P)} \circ (M \otimes p_{N \otimes P}) \circ (i_{M \otimes N} \otimes P) \circ i_{(M \times N) \otimes P} : (M \times N) \times P \rightarrow M \times (N \times P).$$

If $\gamma : M \rightarrow M'$ and $\phi : N \rightarrow N'$ are morphisms in the category, we define

$$\gamma \times \phi = p_{M' \times N'} \circ (\gamma \otimes \phi) \circ i_{M \otimes N} : M \times N \rightarrow M' \times N',$$

that is a morphism in ${}^H_H\mathcal{YD}$ and

$$(\gamma' \times \phi') \circ (\gamma \times \phi) = (\gamma' \circ \gamma) \times (\phi' \circ \phi),$$

where $\gamma' : M' \rightarrow M''$ and $\phi' : N' \rightarrow N''$ are morphisms in ${}^H_H\mathcal{YD}$.

Finally, the braiding is

$$\tau_{M,N} = p_{N \otimes M} \circ t_{M,N} \circ i_{M \otimes N} : M \times N \rightarrow N \times M \quad (8)$$

where

$$t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\varrho_M \otimes N) : M \otimes N \rightarrow N \otimes M. \quad (9)$$

The morphism $\tau_{M,N}$ is a natural isomorphism with inverse:

$$\tau_{M,N}^{-1} = p_{M \otimes N} \circ t'_{M,N} \circ i_{N \otimes M} : N \times M \rightarrow M \times N \quad (10)$$

where

$$t'_{M,N} = c_{N,M} \circ (\varphi_N \otimes M) \circ (c_{N,H} \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \varrho_M). \quad (11)$$

By [[1], Proposition 1.15] we have that given $(M, \varphi_M, \varrho_M)$ in ${}^H_H\mathcal{YD}$, the morphism $t_{M,M} : M \otimes M \rightarrow M \otimes M$ defined in (9) by

$$t_{M,M} = (\varphi_M \otimes M) \circ (H \otimes c_{M,M}) \circ (\varrho_M \otimes M)$$

is a weak Yang-Baxter operator where by (11) we have

$$t'_{M,M} = c_{M,M} \circ (\varphi_M \otimes M) \circ (c_{M,H} \otimes M) \circ (M \otimes \lambda_H^{-1} \otimes M) \circ (M \otimes \varrho_M)$$

and $\nabla_{M,M} = \nabla_{M \otimes M}$.

A similar result can be obtained by working with Yetter-Drinfeld modules associated to a weak Hopf algebra in a braided monoidal category (see [5] for the details).

(2) Let D be in \mathcal{C} where every idempotent morphism splits. If $\Omega : D \otimes D \rightarrow D \otimes D$ is an idempotent morphism such that

$$(\Omega \otimes D) \circ (D \otimes \Omega) = (D \otimes \Omega) \circ (\Omega \otimes D) \quad (12)$$

then Ω is a weak Yang-Baxter operator where $t_{D,D} = t'_{D,D} = \nabla_{D,D} = \Omega$.

Then, as a consequence of (a2-1), the idempotent morphism $\nabla_{D,D}$ of Definition 1.3 is an example of weak Yang-Baxter operator.

It is possible to construct more examples of this kind of weak Yang-Baxter operators working with exact factorizations of groupoids. Previously we recall the definition of wide subgroupoid. A groupoid H is a wide subgroupoid of a groupoid G if H is a subcategory of G provided with a functor $F : H \rightarrow G$ which is the identity on the objects, and it induces inclusions $hom_H(x, y) \subset hom_G(x, y)$, i.e., it has the same base, and (perhaps) less arrows.

Let G be a groupoid. An exact factorization of G is a pair of wide subgroupoids of G , H and V , such that for any $\sigma \in G_1$, there exist unique $\sigma_V \in V_1$, $\sigma_H \in H_1$, such that $\sigma = \sigma_H \circ \sigma_V$.

If G is a groupoid with exact factorization we define

$$\Omega : RG \otimes RG \rightarrow RG \otimes RG$$

as

$$\Omega(\sigma \otimes \tau) = \sigma_H \otimes \tau_V.$$

Then Ω is an idempotent morphism satisfying (12) and then it is a weak Yang-Baxter operator.

(3) In this example we assume that \mathcal{C} is braided. Let D be an algebra in \mathcal{C} . Then the idempotent morphism

$$\Omega = \eta_D \otimes (\mu_D \circ c_{D,D}) : D \otimes D \rightarrow D \otimes D$$

does not satisfy (12) but it is a weak Yang-Baxter operator where

$$t_{D,D} = t'_{D,D} = \nabla_{D,D} = \Omega.$$

Also, if D is a coalgebra in \mathcal{C} , the idempotent morphism

$$\Omega' = \varepsilon_D \otimes (c_{D,D} \circ \delta_D) : D \otimes D \rightarrow D \otimes D$$

is a weak Yang-Baxter operator where

$$t_{D,D} = t'_{D,D} = \nabla_{D,D} = \Omega'.$$

Now we recall the definition of weak braided bialgebra and weak braided Hopf algebra introduced by Alonso, González and Rodríguez in [1] (see also [6]). The interested reader can see the main properties in [[2], Section 2].

Definition 1.5. A weak braided bialgebra (WBB for short) D is an object in \mathcal{C} with an algebra structure (D, η_D, μ_D) and a coalgebra structure $(D, \varepsilon_D, \delta_D)$ such that there exists a weak Yang-Baxter operator $t_{D,D} : D \otimes D \rightarrow D \otimes D$ with associated idempotent $\nabla_{D,D}$ satisfying the following conditions:

(b1) We have

$$(b1-1) \quad \mu_D \circ \nabla_{D,D} = \mu_D,$$

$$(b1-2) \quad \nabla_{D,D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D,D}),$$

$$(b1-3) \quad \nabla_{D,D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D,D} \otimes D).$$

(b2) We have

$$(b2-1) \quad \nabla_{D,D} \circ \delta_D = \delta_D,$$

$$(b2-2) \quad (\delta_D \otimes D) \circ \nabla_{D,D} = (D \otimes \nabla_{D,D}) \circ (\delta_D \otimes D),$$

$$(b2-3) \quad (D \otimes \delta_D) \circ \nabla_{D,D} = (\nabla_{D,D} \otimes D) \circ (D \otimes \delta_D).$$

(b3) The morphisms μ_D and δ_D commute with $t_{D,D}$, i.e.,

$$(b3-1) \quad t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}),$$

$$(b3-2) \quad t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D),$$

$$(b3-3) \quad (\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D),$$

$$(b3-4) \quad (D \otimes \delta_D) \circ t_{D,D} = (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D).$$

(b4) $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D).$

(b5) $\varepsilon_D \circ \mu_D \circ (\mu_D \otimes D) = ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes \delta_D \otimes D)$
 $= ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes (t'_{D,D} \circ \delta_D) \otimes D).$

(b6) $(\delta_D \otimes D) \circ \delta_D \circ \eta_D = (D \otimes \mu_D \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))$
 $= (D \otimes (\mu_D \circ t'_{D,D}) \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D)).$

A weak braided bialgebra D is said to be a weak braided Hopf algebra (WBHA for short) if:

(b7) There exists a morphism $\lambda_D : D \rightarrow D$ in \mathcal{C} (called the antipode of D) satisfying:

$$(b7-1) \quad id_D \wedge \lambda_D = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D),$$

$$(b7-2) \quad \lambda_D \wedge id_D = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)),$$

$$(b7-3) \quad \lambda_D \wedge id_D \wedge \lambda_D = \lambda_D.$$

Let D, B be WBHA. We will say that $f : D \rightarrow B$ is a morphism of WBHA if f is an algebra coalgebra morphism and $t_{B,B} \circ (f \otimes f) = (f \otimes f) \circ t_{D,D}$ and $t'_{B,B} \circ (f \otimes f) = (f \otimes f) \circ t'_{D,D}$.

Examples 1.6. (1) Suppose that \mathcal{C} is symmetric and $t_{D,D} = t'_{D,D}$ is the braiding of the category. Then if D is a WBHA $\nabla_{D,D} = id_{D \otimes D}$ and we obtain the well known definition of weak Hopf algebra [Examples 1.4, (1)].

(2) Now we assume that \mathcal{C} is a braided category with braiding c and $t_{D,D} = c_{D,D}$ and $t'_{D,D} = c_{D,D}^{-1}$. Then $\nabla_{D,D} = id_{D \otimes D}$ and we say that D is a weak Hopf algebra in \mathcal{C} (see [1] and [2]). Obviously, classical Hopf algebras are weak Hopf algebras in this setting and it is not difficult to see that braided Hopf algebras considered by Takeuchi in [21] are examples of weak braided Hopf algebras.

(3) Let the category \mathcal{C} be symmetric and let H be a weak Hopf algebra with invertible antipode in \mathcal{C} . We know [Examples 1.4, (1)] that the category of left-left Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$ is a non-strict braided monoidal category.

An object $(A, \varphi_A, \varrho_A) \in {}^H_H\mathcal{YD}$ is called an algebra if there exist morphisms $u_A : H_L \rightarrow A$ and $m_A : A \times A \rightarrow A$ in ${}^H_H\mathcal{YD}$ such that

$$m_A \circ (u_A \times A) \circ \mathfrak{l}_A^{-1} = id_A = m_A \circ (A \times u_A) \circ \mathfrak{r}_A^{-1}, \quad (13)$$

$$m_A \circ (m_A \times A) \circ \mathfrak{a}_{A,A,A} = m_A \circ (A \times m_A). \quad (14)$$

In a dual way, $(C, \varphi_C, \varrho_C) \in {}^H_H\mathcal{YD}$ is a coalgebra if there exist morphisms $e_C : C \rightarrow H_L$ and $\Delta_C : C \rightarrow C \times C$ in ${}^H_H\mathcal{YD}$ such that

$$\mathfrak{l}_C \circ (e_C \times C) \circ \Delta_C = id_C = \mathfrak{r}_C \circ (C \times e_C) \circ \Delta_C, \quad (15)$$

$$(C \times \Delta_C) \circ \Delta_C = \mathfrak{a}_{C,C,C} \circ (\Delta_C \times C) \circ \Delta_C. \quad (16)$$

A Hopf algebra D in ${}^H_H\mathcal{YD}$ is an algebra-coalgebra in ${}^H_H\mathcal{YD}$, $(D, u_D, m_D, e_D, \Delta_D)$ with a morphism $\lambda_D : D \rightarrow D$ in ${}^H_H\mathcal{YD}$ (called the antipode) such that

$$(i) \quad \Delta_D \circ m_D = (m_D \times m_D) \circ \mathfrak{a}_{D,D,D \times D} \circ (D \times \mathfrak{a}_{D,D,D}^{-1}) \circ (D \times (\tau_{D,D} \times D)) \circ (D \times \mathfrak{a}_{D,D,D}) \circ \mathfrak{a}_{D,D,D \times D}^{-1} \circ (\Delta_D \times \Delta_D),$$

$$(ii) \quad \Delta_D \circ u_D = (u_D \times u_D) \circ \mathfrak{l}_{H_L}^{-1},$$

$$(iii) \quad m_D \circ (D \times \lambda_D) \circ \Delta_D = m_D \circ (\lambda_D \times D) \circ \Delta_D = \mathfrak{r}_D \circ (u_D \times e_D) \circ \mathfrak{l}_D^{-1}.$$

If we define $\eta_D = u_D \circ p_L \circ \eta_H$, $\mu_D = m_D \circ p_{D \otimes D}$, $\varepsilon_D = \varepsilon_H \circ i_L \circ e_D$ and $\delta_D = i_{D \otimes D} \circ \Delta_D$, we have that $(D, \eta_D, \mu_D, \varepsilon_D, \delta_D, \lambda_D)$ is a WBHA in \mathcal{C} [[1], Corollary 2.14]. Note that this example is non-trivial, i.e., D is not a Hopf algebra neither a weak Hopf algebra in the usual sense. For example, if we assume $\varepsilon_D \circ \mu_D = \varepsilon_D \otimes \varepsilon_D$ we obtain that $\Pi_H^L = \varepsilon_H \otimes \eta_H$, or equivalently, H is a Hopf algebra in \mathcal{C} [[1], Remark 2.15]. By an analogous calculus, if $\eta_D \otimes \eta_D = \delta_D \circ \eta_D$, we obtain that H is a Hopf algebra. Moreover, if $\lambda_D \wedge id_D = \varepsilon_D \otimes \eta_D$ we have $u_D \circ e_D = \eta_D \circ \varepsilon_D$ and then $id_{H_L} = p_L \circ \eta_H \circ \varepsilon_H \circ i_L$. Therefore, $\Pi_H^L = \varepsilon_H \otimes \eta_H$ and we obtain that H also is a Hopf algebra. Finally, D is not a weak Hopf algebra since the condition (i) is equivalent to

$$\Delta_D \circ m_D = p_{D,D} \circ (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D) \circ i_{D,D},$$

[[1], Proposition 2.8] and this one does not imply $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes c_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$ where $c_{D,D}$ is the symmetric braiding of \mathcal{C} .

1.7. Let D be a WBB. The following identities hold (see [3])

$$t_{D,D} \circ (\eta_D \otimes D) = \nabla_{D,D} \circ (D \otimes \eta_D) = t'_{D,D} \circ (\eta_D \otimes D), \quad (17)$$

$$t_{D,D} \circ (D \otimes \eta_D) = \nabla_{D,D} \circ (\eta_D \otimes D) = t'_{D,D} \circ (D \otimes \eta_D), \quad (18)$$

$$(D \otimes \varepsilon_D) \circ t_{D,D} = (\varepsilon_D \otimes D) \circ \nabla_{D,D} = (D \otimes \varepsilon_D) \circ t'_{D,D}, \quad (19)$$

$$(\varepsilon_D \otimes D) \circ t_{D,D} = (D \otimes \varepsilon_D) \circ \nabla_{D,D} = (\varepsilon_D \otimes D) \circ t'_{D,D}. \quad (20)$$

Moreover, we have

$$t'_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t'_{D,D} \otimes D) \circ (D \otimes t'_{D,D}), \quad (21)$$

$$t'_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t'_{D,D}) \circ (t'_{D,D} \otimes D), \quad (22)$$

$$(\delta_D \otimes D) \circ t'_{D,D} = (D \otimes t'_{D,D}) \circ (t'_{D,D} \otimes D) \circ (D \otimes \delta_D), \quad (23)$$

$$(D \otimes \delta_D) \circ t'_{D,D} = (t'_{D,D} \otimes D) \circ (D \otimes t'_{D,D}) \circ (\delta_D \otimes D). \quad (24)$$

1.8. Let D be a WBHA. The morphisms Π_D^L (target), Π_D^R (source), $\overline{\Pi}_D^L$ and $\overline{\Pi}_D^R$ are defined as follows:

$$\Pi_D^L = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (t_{D,D} \otimes D) \circ ((\delta_D \circ \eta_D) \otimes D),$$

$$\Pi_D^R = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes t_{D,D}) \circ ((D \otimes (\delta_D \circ \eta_D))),$$

$$\overline{\Pi}_D^L = (D \otimes (\varepsilon_D \circ \mu_D)) \circ ((\delta_D \circ \eta_D) \otimes D),$$

$$\overline{\Pi}_D^R = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)).$$

It is easy to prove that they are idempotent and leave the unit and the counit invariant. Moreover, they satisfy:

$$\Pi_D^L = id_D \wedge \lambda_D, \quad \Pi_D^R = \lambda_D \wedge id_D, \quad \lambda_D = \lambda_D \wedge \Pi_D^L = \Pi_D^R \wedge \lambda_D, \quad (25)$$

and applying (b4) we get

$$id_D \wedge \lambda_D \wedge id_D = \Pi_D^L \wedge id_D = id_D \wedge \Pi_D^R = id_D. \quad (26)$$

Moreover, the following equalities are satisfied [[2], Proposition 2.10]

$$\Pi_D^L \circ \overline{\Pi}_D^L = \Pi_D^L, \quad \Pi_D^L \circ \overline{\Pi}_D^R = \overline{\Pi}_D^R, \quad \overline{\Pi}_D^L \circ \Pi_D^L = \overline{\Pi}_D^L, \quad \overline{\Pi}_D^R \circ \Pi_D^L = \Pi_D^L, \quad (27)$$

$$\Pi_D^R \circ \overline{\Pi}_D^L = \overline{\Pi}_D^L, \quad \Pi_D^R \circ \overline{\Pi}_D^R = \Pi_D^R, \quad \overline{\Pi}_D^L \circ \Pi_D^R = \Pi_D^R, \quad \overline{\Pi}_D^R \circ \Pi_D^R = \overline{\Pi}_D^R, \quad (28)$$

$$\Pi_D^L \circ \lambda_D = \Pi_D^L \circ \Pi_D^R = \lambda_D \circ \Pi_D^R, \quad \Pi_D^R \circ \lambda_D = \Pi_D^R \circ \Pi_D^L = \lambda_D \circ \Pi_D^L, \quad (29)$$

$$\Pi_D^L = \overline{\Pi}_D^R \circ \lambda_D = \lambda_D \circ \overline{\Pi}_D^L, \quad \Pi_D^R = \overline{\Pi}_D^L \circ \lambda_D = \lambda_D \circ \overline{\Pi}_D^R. \quad (30)$$

Finally by [[2], Proposition 2.20] we have that the antipode is antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant, i.e.:

$$\lambda_D \circ \mu_D = \mu_D \circ t_{D,D} \circ (\lambda_D \otimes \lambda_D), \quad (31)$$

$$\delta_D \circ \lambda_D = (\lambda_D \otimes \lambda_D) \circ t_{D,D} \circ \delta_D, \quad (32)$$

$$\lambda_D \circ \eta_D = \eta_D, \quad \varepsilon_D \circ \lambda_D = \varepsilon_D. \quad (33)$$

If $f : D \rightarrow B$ is a morphism of weak braided Hopf algebras, by (2) we obtain $\nabla_{B,B} \circ (f \otimes f) = (f \otimes f) \circ \nabla_{D,D}$. It is not difficult to see that, if $f : D \rightarrow B$ is a morphism of weak braided Hopf algebras, then $f \circ \lambda_D = \lambda_B \circ f$ (see [1] for details).

Once the general framework is stated we introduce the concept of weak operator, that turns out to be essential to define the notion of Yetter-Drinfel'd module in a general monoidal context. Actually, it will allow us to conceive the collection of Yetter-Drinfeld modules as the objects of a monoidal category, being this structure relevant in order to get an operative theory, it is said, a general framework where formal manipulations and effective calculations can be done. It will be obvious from the definition below that weak operators constitute a generalization of the notion of weak Yang-Baxter operator.

Definition 1.9. Let D be a WBHA and let M be an object of \mathcal{C} . A weak operator between M and D , (from now on referred as (M, D) -WO) is defined as a quadruple (r_M, r'_M, s_M, s'_M) comprised of four morphisms in \mathcal{C} :

$$r_M : M \otimes D \rightarrow D \otimes M, \quad r'_M : D \otimes M \rightarrow M \otimes D,$$

$$s_M : D \otimes M \rightarrow M \otimes D, \quad s'_M : M \otimes D \rightarrow D \otimes M,$$

satisfying the following compatibility conditions:

(c1) Compatibility with the weak Yang-Baxter operator:

$$\begin{aligned} \text{(c1-1)} \quad & (D \otimes r_M) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}) = (t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (r_M \otimes D), \\ \text{(c1-2)} \quad & (r'_M \otimes D) \circ (D \otimes r'_M) \circ (t_{D,D} \otimes M) = (M \otimes t_{D,D}) \circ (r'_M \otimes D) \circ (D \otimes r'_M), \\ \text{(c1-3)} \quad & (s_M \otimes D) \circ (D \otimes s_M) \circ (t_{D,D} \otimes M) = (M \otimes t_{D,D}) \circ (s_M \otimes D) \circ (D \otimes s_M), \\ \text{(c1-4)} \quad & (D \otimes s'_M) \circ (s'_M \otimes D) \circ (M \otimes t_{D,D}) = (t_{D,D} \otimes M) \circ (D \otimes s'_M) \circ (s'_M \otimes D). \end{aligned}$$

The analogous equalities with $t'_{D,D}$ instead of $t_{D,D}$ are also required to be satisfied.

(c2) Mixed Yang-Baxter equations

$$\begin{aligned} \text{(c2-1)} \quad & (r'_M \otimes D) \circ (D \otimes s_M) \circ (t_{D,D} \otimes M) = (M \otimes t_{D,D}) \circ (s_M \otimes D) \circ (D \otimes r'_M), \\ \text{(c2-2)} \quad & (s_M \otimes D) \circ (D \otimes r'_M) \circ (t'_{D,D} \otimes M) = (M \otimes t'_{D,D}) \circ (r'_M \otimes D) \circ (D \otimes s_M), \\ \text{(c2-3)} \quad & (D \otimes s'_M) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}) = (t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (s'_M \otimes D), \\ \text{(c2-4)} \quad & (D \otimes r_M) \circ (s'_M \otimes D) \circ (M \otimes t'_{D,D}) = (t'_{D,D} \otimes M) \circ (D \otimes s'_M) \circ (r_M \otimes D). \end{aligned}$$

We want to point out that in this case, as in general for all the mixed equations along the paper, we cannot replace $t_{D,D}$ by $t'_{D,D}$ or $t'_{D,D}$ by $t_{D,D}$.

(c3) The morphisms: $\nabla_{r_M} := r'_M \circ r_M$, $\nabla_{r'_M} := r_M \circ r'_M$, $\nabla_{s_M} := s'_M \circ s_M$ and $\nabla_{s'_M} := s_M \circ s'_M$ satisfy

$$\begin{aligned} \text{(c3-1)} \quad & \nabla_{r_M} = (((\varepsilon_D \otimes M) \circ r_M) \otimes D) \circ (M \otimes \delta_D) = (M \otimes \mu_D) \circ (((r'_M \circ (\eta_D \otimes M)) \otimes D), \\ \text{(c3-2)} \quad & \nabla_{r'_M} = (D \otimes ((M \otimes \varepsilon_D) \circ r'_M)) \circ (\delta_D \otimes M) = (\mu_D \otimes M) \circ (D \otimes (r_M \circ (M \otimes \eta_D))), \\ \text{(c3-3)} \quad & \nabla_{s_M} = (D \otimes ((M \otimes \varepsilon_D) \circ s_M)) \circ (\delta_D \otimes M) = (\mu_D \otimes M) \circ (D \otimes (s'_M \circ (M \otimes \eta_D))), \\ \text{(c3-4)} \quad & \nabla_{s'_M} = (((\varepsilon_D \otimes M) \circ s'_M) \otimes D) \circ (M \otimes \delta_D) = (M \otimes \mu_D) \circ (((s_M \circ (\eta_D \otimes M)) \otimes D). \end{aligned}$$

(c4) Compatibility with the (co) multiplication:

$$\begin{aligned} \text{(c4-1)} \quad & r_M \circ (M \otimes \mu_D) = (\mu_D \otimes M) \circ (D \otimes r_M) \circ (r_M \otimes D), \\ \text{(c4-2)} \quad & r'_M \circ (\mu_D \otimes M) = (M \otimes \mu_D) \circ (r'_M \otimes D) \circ (D \otimes r'_M), \\ \text{(c4-3)} \quad & (D \otimes r_M) \circ (r_M \otimes D) \circ (M \otimes \delta_D) = (\delta_D \otimes M) \circ r_M, \\ \text{(c4-4)} \quad & (r'_M \otimes D) \circ (D \otimes r'_M) \circ (\delta_D \otimes M) = (M \otimes \delta_D) \circ r'_M, \\ \text{(c4-5)} \quad & s_M \circ (\mu_D \otimes M) = (M \otimes \mu_D) \circ (s_M \otimes D) \circ (D \otimes s_M), \\ \text{(c4-6)} \quad & s'_M \circ (M \otimes \mu_D) = (\mu_D \otimes M) \circ (D \otimes s'_M) \circ (s'_M \otimes D), \\ \text{(c4-7)} \quad & (s_M \otimes D) \circ (D \otimes s_M) \circ (\delta_D \otimes M) = (M \otimes \delta_D) \circ s_M, \\ \text{(c4-8)} \quad & (D \otimes s'_M) \circ (s'_M \otimes D) \circ (M \otimes \delta_D) = (\delta_D \otimes M) \circ s'_M. \end{aligned}$$

(c5) Compatibility with the antipode:

$$\begin{aligned} \text{(c5-1)} \quad & (M \otimes \lambda_D) \circ \nabla_{r_M} = \nabla_{r_M} \circ (M \otimes \lambda_D), \\ \text{(c5-2)} \quad & (\lambda_D \otimes M) \circ \nabla_{r'_M} = \nabla_{r'_M} \circ (\lambda_D \otimes M), \\ \text{(c5-3)} \quad & (\lambda_D \otimes M) \circ \nabla_{s_M} = \nabla_{s_M} \circ (\lambda_D \otimes M), \\ \text{(c5-4)} \quad & (M \otimes \lambda_D) \circ \nabla_{s'_M} = \nabla_{s'_M} \circ (M \otimes \lambda_D). \end{aligned}$$

Remark 1.10. As a consequence of Definition 1.9 the following equalities hold for a WBHA D :

$$(D \otimes r_M) \circ (r_M \otimes D) \circ (M \otimes \nabla_{D,D}) = (\nabla_{D,D} \otimes M) \circ (D \otimes r_M) \circ (r_M \otimes D), \quad (34)$$

$$(r'_M \otimes D) \circ (D \otimes r'_M) \circ (\nabla_{D,D} \otimes M) = (M \otimes \nabla_{D,D}) \circ (r'_M \otimes D) \circ (D \otimes r'_M), \quad (35)$$

$$(D \otimes r_M) \circ (s'_M \otimes D) \circ (M \otimes \nabla_{D,D}) = (\nabla_{D,D} \otimes M) \circ (D \otimes r_M) \circ (s'_M \otimes D), \quad (36)$$

$$(s_M \otimes D) \circ (D \otimes r'_M) \circ (\nabla_{D,D} \otimes M) = (M \otimes \nabla_{D,D}) \circ (s_M \otimes D) \circ (D \otimes r'_M), \quad (37)$$

$$(M \otimes \varepsilon_D) \circ \nabla_{r_M} = (\varepsilon_D \otimes M) \circ r_M; \quad (\varepsilon_D \otimes M) \circ \nabla_{r'_M} = (M \otimes \varepsilon_D) \circ r'_M, \quad (38)$$

$$\nabla_{r_M} \circ (M \otimes \eta_D) = r'_M \circ (\eta_D \otimes M); \quad \nabla_{r'_M} \circ (\eta_D \otimes M) = r_M \circ (M \otimes \eta_D), \quad (39)$$

$$\nabla_{r_M} \circ (M \otimes \mu_D) = (M \otimes \mu_D) \circ (\nabla_{r_M} \otimes D), \quad (40)$$

$$\nabla_{r'_M} \circ (\mu_D \otimes M) = (\mu_D \otimes M) \circ (D \otimes \nabla_{r'_M}), \quad (41)$$

$$(\nabla_{r_M} \otimes D) \circ (M \otimes \delta_D) = (M \otimes \delta_D) \circ \nabla_{r_M}, \quad (42)$$

$$(D \otimes \nabla_{r'_M}) \circ (\delta_D \otimes M) = (\delta_D \otimes M) \circ \nabla_{r'_M}, \quad (43)$$

$$(D \otimes \nabla_{r_M}) \circ (r_M \otimes D) \circ (M \otimes \delta_D) = (r_M \otimes D) \circ (M \otimes \delta_D), \quad (44)$$

$$(\nabla_{r'_M} \otimes D) \circ (D \otimes r'_M) \circ (\delta_D \otimes M) = (D \otimes r'_M) \circ (\delta_D \otimes M), \quad (45)$$

$$(s_M \otimes D) \circ (D \otimes s_M) \circ (\nabla_{D,D} \otimes M) = (M \otimes \nabla_{D,D}) \circ (s_M \otimes D) \circ (D \otimes s_M), \quad (46)$$

$$(D \otimes s'_M) \circ (s'_M \otimes D) \circ (M \otimes \nabla_{D,D}) = (\nabla_{D,D} \otimes M) \circ (D \otimes s'_M) \circ (s'_M \otimes D), \quad (47)$$

$$(r'_M \otimes D) \circ (D \otimes s_M) \circ (\nabla_{D,D} \otimes M) = (M \otimes \nabla_{D,D}) \circ (r'_M \otimes D) \circ (D \otimes s_M), \quad (48)$$

$$(D \otimes s'_M) \circ (r_M \otimes D) \circ (M \otimes \nabla_{D,D}) = (\nabla_{D,D} \otimes M) \circ (D \otimes s'_M) \circ (r_M \otimes D), \quad (49)$$

$$(\varepsilon_D \otimes M) \circ \nabla_{s_M} = (M \otimes \varepsilon_D) \circ s_M; \quad (M \otimes \varepsilon_D) \circ \nabla_{s'_M} = (\varepsilon_D \otimes M) \circ s'_M, \quad (50)$$

$$\nabla_{s'_M} \circ (M \otimes \eta_D) = s_M \circ (\eta_D \otimes M); \quad \nabla_{s_M} \circ (\eta_D \otimes M) = s'_M \circ (M \otimes \eta_D), \quad (51)$$

$$\nabla_{s_M} \circ (\mu_D \otimes M) = (\mu_D \otimes M) \circ (D \otimes \nabla_{s_M}), \quad (52)$$

$$\nabla_{s'_M} \circ (M \otimes \mu_D) = (M \otimes \mu_D) \circ (\nabla_{s'_M} \otimes D), \quad (53)$$

$$(D \otimes s_M) \circ (\delta_D \otimes M) = (\delta_D \otimes M) \circ \nabla_{s_M}, \quad (54)$$

$$(\nabla_{s'_M} \otimes D) \circ (M \otimes \delta_D) = (M \otimes \delta_D) \circ \nabla_{s'_M}, \quad (55)$$

$$(D \otimes \nabla_{s'_M}) \circ (s'_M \otimes D) \circ (M \otimes \delta_D) = (s'_M \otimes D) \circ (M \otimes \delta_D), \quad (56)$$

$$(\nabla_{s_M} \otimes D) \circ (D \otimes s_M) \circ (\delta_D \otimes M) = (D \otimes s_M) \circ (\delta_D \otimes M). \quad (57)$$

Note that (r_M, r'_M, s_M, s'_M) constitutes an (M, D) -WO iff so does (s'_M, s_M, r'_M, r_M) .

Proposition 1.11. Let D be a WBHA and let M be an object of \mathcal{C} such that (r_M, r'_M, s_M, s'_M) constitutes an (M, D) -WO. Then it holds that:

- (i) The morphisms ∇_{r_M} , $\nabla_{r'_M}$, ∇_{s_M} and $\nabla_{s'_M}$ are idempotent.
- (ii) Cancellation laws:

$$r_M = \nabla_{r'_M} \circ r_M = r_M \circ \nabla_{r_M}, \quad (58)$$

$$r'_M = r'_M \circ \nabla_{r'_M} = \nabla_{r_M} \circ r'_M, \quad (59)$$

$$s_M = \nabla_{s'_M} \circ s_M = s_M \circ \nabla_{s_M}, \quad (60)$$

$$s'_M = s'_M \circ \nabla_{s'_M} = \nabla_{s_M} \circ s'_M. \quad (61)$$

Proof. For (i):

$$\begin{aligned} & \nabla_{r_M} \circ \nabla_{r_M} \\ &= (((\varepsilon_D \otimes M) \circ r_M) \otimes D) \circ (((\varepsilon_D \otimes M) \circ r_M) \otimes \delta_D) \circ (M \otimes \delta_D) \\ &= (\varepsilon_D \otimes \varepsilon_D \otimes M \otimes D) \circ (\delta_D \otimes M \otimes D) \circ (r_M \otimes D) \circ (M \otimes \delta_D) \\ &= \nabla_{r_M} \end{aligned}$$

by coassociativity and (c4-3). The proofs for the remaining morphisms are analogous.

To prove (ii) it suffices to use the suitable characterization of the corresponding morphisms and then apply the compatibility with the (co)multiplication. We write the first equality of (58) to illustrate the procedure:

$$\begin{aligned} & r_M \\ &= r_M \circ (M \otimes (\mu_D \circ (D \otimes \eta_D))) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (r_M \otimes \eta_D) \\ &= \nabla_{r'_M} \circ r_M \end{aligned}$$

□

Proposition 1.12. Let D be a WBHA, M any object of the category and (r_M, r'_M, s_M, s'_M) an (M, D) -WO. Then we have:

$$(D \otimes \nabla_{r_M}) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}) = (r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (\nabla_{r_M} \otimes D), \quad (62)$$

$$(t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (\nabla_{r'_M} \otimes D) = (D \otimes \nabla_{r'_M}) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M), \quad (63)$$

$$(\nabla_{r'_M} \otimes D) \circ (D \otimes r'_M) \circ (t_{D,D} \otimes M) = (D \otimes r'_M) \circ (t_{D,D} \otimes M) \circ (D \otimes \nabla_{r'_M}), \quad (64)$$

$$(M \otimes t_{D,D}) \circ (r'_M \otimes D) \circ (D \otimes \nabla_{r_M}) = (\nabla_{r_M} \otimes D) \circ (M \otimes t_{D,D}) \circ (r'_M \otimes D), \quad (65)$$

$$(\nabla_{s_M} \otimes D) \circ (D \otimes s_M) \circ (t_{D,D} \otimes M) = (D \otimes s_M) \circ (t_{D,D} \otimes M) \circ (D \otimes \nabla_{s_M}), \quad (66)$$

$$(M \otimes t_{D,D}) \circ (s_M \otimes D) \circ (D \otimes \nabla_{s'_M}) = (\nabla_{s'_M} \otimes D) \circ (M \otimes t_{D,D}) \circ (s_M \otimes D), \quad (67)$$

$$(D \otimes \nabla_{s'_M}) \circ (s'_M \otimes D) \circ (M \otimes t_{D,D}) = (s'_M \otimes D) \circ (M \otimes t_{D,D}) \circ (\nabla_{s'_M} \otimes D), \quad (68)$$

$$(t_{D,D} \otimes M) \circ (D \otimes s'_M) \circ (\nabla_{s_M} \otimes D) = (D \otimes \nabla_{s_M}) \circ (t_{D,D} \otimes M) \circ (D \otimes s'_M). \quad (69)$$

The equalities remain true if we change $t_{D,D}$ by $t'_{D,D}$.

Proof. For (62);

$$\begin{aligned} & (D \otimes \nabla_{r_M}) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}) \\ &= (D \otimes ((\varepsilon_D \otimes M) \circ r_M) \otimes D) \circ (r_M \otimes \delta_D) \circ (M \otimes t_{D,D}) \\ &= (D \otimes ((\varepsilon_D \otimes M) \circ r_M) \otimes D) \circ (r_M \otimes D \otimes D) \circ (M \otimes t_{D,D} \otimes D) \circ (M \otimes D \otimes t_{D,D}) \\ &\quad \circ (M \otimes \delta_D \otimes D) \\ &= (((D \otimes \varepsilon_D) \circ t_{D,D}) \otimes M \otimes D) \circ (D \otimes r_M \otimes D) \circ (r_M \otimes t_{D,D}) \circ (M \otimes \delta_D \otimes D) \\ &= (((\varepsilon_D \otimes D) \circ \nabla_{D,D}) \otimes M \otimes D) \circ (D \otimes r_M \otimes D) \circ (r_M \otimes t_{D,D}) \circ (M \otimes \delta_D \otimes D) \\ &= (\varepsilon_D \otimes r_M \otimes D) \circ (r_M \otimes D \otimes D) \circ (M \otimes \nabla_{D,D} \otimes D) \circ (M \otimes D \otimes t_{D,D}) \circ (M \otimes \delta_D \otimes D) \\ &= (\varepsilon_D \otimes r_M \otimes D) \circ (r_M \otimes t_{D,D}) \circ (M \otimes \delta_D \otimes D) \\ &= (r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (\nabla_{r_M} \otimes D), \end{aligned}$$

where we used (c3-1), the conditions (b3-4) and (c1-1), the properties of the weak Yang-Baxter operator and the equalities (19) and (34).

The proof of the remaining equalities follows a similar procedure. \square

Proposition 1.13. Let D be a WBHA, M any object of the category and (r_M, r'_M, s_M, s'_M) an (M, D) -WO. Then we have:

$$(t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (\nabla_{s_M} \otimes D) = (D \otimes \nabla_{s_M}) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M), \quad (70)$$

$$(r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (\nabla_{s'_M} \otimes D) = (D \otimes \nabla_{s'_M}) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}), \quad (71)$$

$$(s'_M \otimes D) \circ (M \otimes t'_{D,D}) \circ (\nabla_{r_M} \otimes D) = (D \otimes \nabla_{r_M}) \circ (s'_M \otimes D) \circ (M \otimes t'_{D,D}), \quad (72)$$

$$(t'_{D,D} \otimes M) \circ (D \otimes s'_M) \circ (\nabla_{r'_M} \otimes D) = (D \otimes \nabla_{r'_M}) \circ (t'_{D,D} \otimes M) \circ (D \otimes s'_M). \quad (73)$$

Proof. We will show (70):

$$\begin{aligned} & (t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (\nabla_{s_M} \otimes D) \\ &= (D \otimes \mu_D \otimes M) \circ (t_{D,D} \otimes D \otimes M) \circ (D \otimes t_{D,D} \otimes M) \circ (D \otimes D \otimes r_M) \circ (D \otimes (s'_M \circ (M \otimes \eta_D)) \otimes D) \\ &= (D \otimes \mu_D \otimes M) \circ (t_{D,D} \otimes s'_M) \circ (D \otimes r_M \otimes D) \circ (D \otimes M \otimes (t_{D,D} \circ (\eta_D \otimes D))) \\ &= (D \otimes \mu_D \otimes M) \circ (t_{D,D} \otimes s'_M) \circ (D \otimes r_M \otimes D) \circ (D \otimes M \otimes (\nabla_{D,D} \circ (D \otimes \eta_D))) \\ &= (D \otimes \mu_D \otimes M) \circ (t_{D,D} \otimes D \otimes M) \circ (D \otimes \nabla_{D,D} \otimes M) \circ (D \otimes D \otimes (s'_M \circ (M \otimes \eta_D))) \circ (D \otimes r_M) \\ &= (D \otimes \nabla_{s_M}) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M), \end{aligned}$$

where we used (c3-4), the conditions (b1-1) and (c2-3), the properties of the weak Yang-Baxter operator and the equalities (17) and (49). \square

Proposition 1.14. Let D be a WBHA, M any object of the category and (r_M, r'_M, s_M, s'_M) an (M, D) -WO. Then it holds that:

(i)

$$(r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (r'_M \otimes D) = (D \otimes r'_M) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M), \quad (74)$$

$$(D \otimes s_M) \circ (t_{D,D} \otimes M) \circ (D \otimes s'_M) = (s'_M \otimes D) \circ (M \otimes t_{D,D}) \circ (s_M \otimes D). \quad (75)$$

The previous equalities remain true if we change $t_{D,D}$ by $t'_{D,D}$.

(ii)

$$(r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (s_M \otimes D) = (D \otimes s_M) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M), \quad (76)$$

$$(s'_M \otimes D) \circ (M \otimes t'_{D,D}) \circ (r'_M \otimes D) = (D \otimes r'_M) \circ (t'_{D,D} \otimes M) \circ (D \otimes s'_M). \quad (77)$$

Proof. We prove the first equality of (i), the remaining being analogous:

$$\begin{aligned} & (r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (r'_M \otimes D) \\ &= (r_M \otimes D) \circ (M \otimes t_{D,D}) \circ ((\nabla_{r_M} \circ r'_M) \otimes D) \\ &= (D \otimes \nabla_{r_M}) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (r'_M \otimes D) \\ &= (D \otimes r'_M) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M) \circ ((r_M \circ r'_M) \otimes D) \\ &= (D \otimes r'_M) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (\nabla_{r'_M} \otimes D) \\ &= (D \otimes (r'_M \circ \nabla_{r'_M})) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M) \\ &= (D \otimes r'_M) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M), \end{aligned}$$

In the above equalities, the first and the last ones follow by part (ii) of Proposition 1.11, the second and the fifth by (62) and (63), respectively. In the third we use (c1-1) and the fourth follows by the definition of $\nabla_{r'_M}$.

The proof of (ii) is analogous to the one of (i) but applying (70) and (71) instead of (62) and (63) and the condition (c2-3). \square

Remark 1.15. In view of Definition 1.9, it follows that if $M = D$ is a WBHA in \mathcal{C} , the associated weak Yang-Baxter operator $t_{D,D}$ is an example of (D, D) -WO with $r_M = s_M = t_{D,D}$ and $r'_M = s'_M = t'_{D,D}$; the claim remaining true if we take $t'_{D,D}$ instead of $t_{D,D}$ and vice versa.

Of course if $(\mathcal{C}, \otimes, c)$ is a braided monoidal category, the quadruples $(c_{M,D}, c_{M,D}^{-1}, c_{D,M}, c_{D,M}^{-1})$ and $(c_{D,M}^{-1}, c_{D,M}, c_{M,D}^{-1}, c_{M,D})$ are examples of (M, D) -WO for any object M of \mathcal{C} .

Moreover, going into the interpretation of the notion of weak operator as a generalization of that of weak Yang-Baxter operator we point out the following series of results (See [2]).

Proposition 1.16. With the assumptions and notation of Proposition 1.14, we have:

$$(r_M \otimes D) \circ (M \otimes (\delta_D \circ \eta_D)) = (D \otimes r'_M) \circ ((\delta_D \circ \eta_D) \otimes M), \quad (78)$$

$$((\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes r_M) = (M \otimes (\varepsilon_D \circ \mu_D)) \circ (r'_M \otimes D), \quad (79)$$

$$(s'_M \otimes D) \circ (M \otimes (\delta_D \circ \eta_D)) = (D \otimes s_M) \circ ((\delta_D \circ \eta_D) \otimes M), \quad (80)$$

$$((\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes s'_M) = (M \otimes (\varepsilon_D \circ \mu_D)) \circ (s_M \otimes D). \quad (81)$$

Proof. We prove (78), the others being analogous:

$$\begin{aligned} & (r_M \otimes D) \circ (M \otimes (\delta_D \circ \eta_D)) \\ &= ((r_M \circ \nabla_{r_M}) \otimes D) \circ (M \otimes (\delta_D \circ \eta_D)) \\ &= (r_M \otimes D) \circ (M \otimes \delta_D) \circ \nabla_{r_M} \circ (M \otimes \eta_D) \\ &= (r_M \otimes D) \circ (M \otimes \delta_D) \circ r'_M \circ (\eta_D \otimes M) \\ &= ((r_M \circ r'_M) \otimes D) \circ (D \otimes r'_M) \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= (D \otimes ((M \otimes \varepsilon_D) \circ r'_M) \otimes D) \circ (\delta_D \otimes M \otimes D) \circ (D \otimes r'_M) \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= (D \otimes r'_M) \circ ((\delta_D \circ \eta_D) \otimes M). \end{aligned}$$

In the above equalities we use that D is a coalgebra, part (ii) of Proposition 1.11, the conditions (c3-1), (c3-2) and (c4-4), and the equality (42). \square

Proposition 1.17. Let D be a WBHA and M any object in \mathcal{C} . If (r_M, r'_M, s_M, s'_M) is an (M, D) -WO, it holds that:

$$(M \otimes \Pi_D^L) \circ \nabla_{r_M} = \nabla_{r_M} \circ (M \otimes \Pi_D^L), \quad (82)$$

$$(\Pi_D^L \otimes M) \circ \nabla_{r'_M} = \nabla_{r'_M} \circ (\Pi_D^L \otimes M), \quad (83)$$

$$(\Pi_D^L \otimes M) \circ \nabla_{s_M} = \nabla_{s_M} \circ (\Pi_D^L \otimes M), \quad (84)$$

$$(M \otimes \Pi_D^L) \circ \nabla_{s'_M} = \nabla_{s'_M} \circ (M \otimes \Pi_D^L). \quad (85)$$

The analogous equalities hold writing either Π_D^R , $\bar{\Pi}_D^L$, or $\bar{\Pi}_D^R$ instead of Π_D^L .

Proof. We prove (82) and (83), being the others analogous. Applying the definition of Π_D^L and the equalities (40) and (42) we have:

$$\begin{aligned} & (M \otimes \Pi_D^L) \circ \nabla_{r_M} \\ &= (M \otimes \mu_D) \circ (\nabla_{r_M} \otimes \lambda_D) \circ (M \otimes \delta_D) \\ &= \nabla_{r_M} \circ (M \otimes (\mu_D \circ (D \otimes \lambda_D) \circ \delta_D)) \\ &= \nabla_{r_M} \circ (M \otimes \Pi_D^L). \end{aligned}$$

Now by the definition of Π_D^L , the condition (c5-2) and the equalities (41) and (43) we get:

$$\begin{aligned} & (\Pi_D^L \otimes M) \circ \nabla_{r'_M} \\ &= (\mu_D \otimes M) \circ (D \otimes \lambda_D \otimes M) \circ (D \otimes \nabla_{r'_M}) \circ (\delta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes \nabla_{r'_M}) \circ (D \otimes \lambda_D \otimes M) \circ (\delta_D \otimes M) \\ &= \nabla_{r'_M} \circ ((\mu_D \circ (D \otimes \lambda_D) \circ \delta_D) \otimes M) \\ &= \nabla_{r'_M} \circ (\Pi_D^L \otimes M). \end{aligned}$$

Analogously we prove:

$$\nabla_{r_M} \circ (M \otimes \Pi_D^R) = (M \otimes \Pi_D^R) \circ \nabla_{r_M} \text{ and } \nabla_{r'_M} \circ (\Pi_D^R \otimes M) = (\Pi_D^R \otimes M) \circ \nabla_{r'_M}.$$

It is now easy to prove the corresponding equalities for $\bar{\Pi}_D^L$ and $\bar{\Pi}_D^R$ just using (27) and (28). \square

Proposition 1.18. Let D be a WBHA and M any object in \mathcal{C} . If (r_M, r'_M, s_M, s'_M) is an (M, D) -WO, it holds that:

$$(\Pi_D^L \otimes M) \circ r_M = r_M \circ (M \otimes \Pi_D^L), \quad (86)$$

$$(M \otimes \Pi_D^L) \circ r'_M = r'_M \circ (\Pi_D^L \otimes M), \quad (87)$$

$$(M \otimes \Pi_D^L) \circ s_M = s_M \circ (\Pi_D^L \otimes M), \quad (88)$$

$$(\Pi_D^L \otimes M) \circ s'_M = s'_M \circ (M \otimes \Pi_D^L). \quad (89)$$

The analogous equalities hold writing either Π_D^R , $\bar{\Pi}_D^L$, or $\bar{\Pi}_D^R$ instead of Π_D^L .

Proof. We will show (86). Firstly note that

$$\begin{aligned} & (M \otimes \Pi_D^L) \circ \nabla_{r_M} \\ &= ((\varepsilon_D \otimes M) \circ r_M \otimes \Pi_D^L) \circ (M \otimes \delta_D) \\ &= (((\varepsilon_D \otimes M) \circ r_M) \otimes D) \circ (M \otimes \mu_D \otimes D) \circ (M \otimes D \otimes t_{D,D}) \circ (M \otimes (\delta_D \circ \eta_D) \otimes D) \\ &= ((\varepsilon_D \circ \mu_D) \otimes M \otimes D) \circ (D \otimes r_M \otimes D) \circ (r_M \otimes t_{D,D}) \circ (M \otimes (\delta_D \circ \eta_D) \otimes D) \\ &= ((\varepsilon_D \circ \mu_D) \otimes M \otimes D) \circ (D \otimes r_M \otimes D) \circ (D \otimes M \otimes t_{D,D}) \circ (D \otimes r'_M \otimes D) \\ &\quad \circ ((\delta_D \circ \eta_D) \otimes M \otimes D) \\ &= ((\varepsilon_D \circ \mu_D) \otimes r'_M) \circ (D \otimes t_{D,D} \otimes M) \circ ((\delta_D \circ \eta_D) \otimes r_M) \\ &= r'_M \circ (\Pi_D^L \otimes M) \circ r_M. \end{aligned}$$

In the above calculations, we applied (c3-1), the equality

$$(D \otimes \Pi_D^L) \circ \delta_D = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D), \quad (90)$$

the condition (c4-1) and the equalities (78) and (74).

Hence by (82) it holds that:

$$(M \otimes \Pi_D^L) \circ \nabla_{r_M} = r'_M \circ (\Pi_D^L \otimes M) \circ r_M = \nabla_{r_M} \circ (M \otimes \Pi_D^L). \quad (91)$$

Now, applying the definition of $\nabla_{r'_M}$, the equality (83) and part (ii) of Proposition 1.11 we get:

$$\begin{aligned} & (\Pi_D^L \otimes M) \circ r_M \\ &= (\Pi_D^L \otimes M) \circ \nabla_{r'_M} \circ r_M \\ &= \nabla_{r'_M} \circ (\Pi_D^L \otimes M) \circ r_M \\ &= r_M \circ r'_M \circ (\Pi_D^L \otimes M) \circ r_M \\ &= r_M \circ \nabla_{r_M} \circ (M \otimes \Pi_D^L) \\ &= r_M \circ (M \otimes \Pi_D^L). \end{aligned}$$

□

Proposition 1.19. In the hypothesis of Proposition 1.18, it holds that:

$$(\lambda_D \otimes M) \circ r_M = r_M \circ (M \otimes \lambda_D), \quad (92)$$

$$(M \otimes \lambda_D) \circ r'_M = r'_M \circ (\lambda_D \otimes M), \quad (93)$$

$$(M \otimes \lambda_D) \circ s_M = s_M \circ (\lambda_D \otimes M), \quad (94)$$

$$(\lambda_D \otimes M) \circ s'_M = s'_M \circ (M \otimes \lambda_D). \quad (95)$$

If λ_D is an isomorphism all the corresponding equalities obtained writing λ_D^{-1} instead of λ_D are also verified.

Proof. To deduce (92) we can write:

$$\begin{aligned} & (\lambda_D \otimes M) \circ r_M \\ &= (\lambda_D \wedge \Pi_D^L \otimes M) \circ r_M \\ &= ((\mu_D \circ (\lambda_D \otimes \Pi_D^L)) \otimes M) \circ (D \otimes r_M) \circ (r_M \otimes D) \circ (M \otimes \delta_D) \\ &= (\mu_D \otimes M) \circ (\lambda_D \otimes r_M) \circ (r_M \otimes \Pi_D^L) \circ (M \otimes \delta_D) \\ &= (\mu_D \otimes M) \circ (\lambda_D \otimes r_M) \circ (r_M \otimes \mu_D) \circ (M \otimes \delta_D \otimes \lambda_D) \circ (M \otimes \delta_D) \\ &= (\mu_D \otimes M) \circ (\mu_D \otimes r_M) \circ (\lambda_D \otimes r_M \otimes D) \circ (r_M \otimes D \otimes \lambda_D) \circ (M \otimes \delta_D \otimes D) \circ (M \otimes \delta_D) \\ &= (\mu_D \otimes M) \circ (\Pi_D^R \otimes r_M) \circ (r_M \otimes \lambda_D) \circ (M \otimes \delta_D) \\ &= r_M \circ (M \otimes \Pi_D^R \wedge \lambda_D) \\ &= r_M \circ (M \otimes \lambda_D), \end{aligned}$$

In the preceding calculations, the first, fourth and eighth equalities rely on the definition of WBB, the second, fifth and sixth on (c4), and the third and seventh ones follow by Proposition 1.18.

In a similar way we obtain the equality for r'_M , s_M and s'_M . Finally, by composing with λ_D^{-1} we get the similar equalities involving the inverse of the antipode. \square

Corollary 1.20. Let D be a WBHA with invertible antipode and M any object in \mathcal{C} . If (r_M, r'_M, s_M, s'_M) is an (M, D) -WO, the following equalities hold:

(i)

$$\nabla_{r_M} = (M \otimes (\mu_D \circ t_{D,D})) \circ (r'_M \otimes D) \circ (\eta_D \otimes M \otimes D), \quad (96)$$

$$\nabla_{r_M} = (M \otimes (\mu_D \circ t'_{D,D})) \circ (r'_M \otimes D) \circ (\eta_D \otimes M \otimes D), \quad (97)$$

$$\nabla_{r_M} = (\varepsilon_D \otimes M \otimes D) \circ (r_M \otimes D) \circ (M \otimes (t_{D,D} \circ \delta_D)), \quad (98)$$

$$\nabla_{r_M} = (\varepsilon_D \otimes M \otimes D) \circ (r_M \otimes D) \circ (M \otimes (t'_{D,D} \circ \delta_D)). \quad (99)$$

(ii)

$$\nabla_{r'_M} = ((\mu_D \circ t_{D,D}) \otimes M) \circ (D \otimes r_M) \circ (D \otimes M \otimes \eta_D), \quad (100)$$

$$\nabla_{r'_M} = ((\mu_D \circ t'_{D,D}) \otimes M) \circ (D \otimes r_M) \circ (D \otimes M \otimes \eta_D), \quad (101)$$

$$\nabla_{r'_M} = (D \otimes M \otimes \varepsilon_D) \circ (D \otimes r'_M) \circ ((t_{D,D} \circ \delta_D) \otimes M), \quad (102)$$

$$\nabla_{r'_M} = (D \otimes M \otimes \varepsilon_D) \circ (D \otimes r'_M) \circ ((t'_{D,D} \circ \delta_D) \otimes M). \quad (103)$$

(iii)

$$\nabla_{s_M} = ((\mu_D \circ t_{D,D}) \otimes M) \circ (D \otimes s'_M) \circ (D \otimes M \otimes \eta_D), \quad (104)$$

$$\nabla_{s_M} = ((\mu_D \circ t'_{D,D}) \otimes M) \circ (D \otimes s'_M) \circ (D \otimes M \otimes \eta_D), \quad (105)$$

$$\nabla_{s_M} = (D \otimes M \otimes \varepsilon_D) \circ (D \otimes s_M) \circ ((t_{D,D} \circ \delta_D) \otimes M), \quad (106)$$

$$\nabla_{s_M} = (D \otimes M \otimes \varepsilon_D) \circ (D \otimes s_M) \circ ((t'_{D,D} \circ \delta_D) \otimes M). \quad (107)$$

(iv)

$$\nabla_{s'_M} = (M \otimes (\mu_D \circ t_{D,D})) \circ (s_M \otimes D) \circ (\eta_D \otimes M \otimes D), \quad (108)$$

$$\nabla_{s'_M} = (M \otimes (\mu_D \circ t'_{D,D})) \circ (s_M \otimes D) \circ (\eta_D \otimes M \otimes D), \quad (109)$$

$$\nabla_{s'_M} = (\varepsilon_D \otimes M \otimes D) \circ (s'_M \otimes D) \circ (M \otimes (t_{D,D} \circ \delta_D)), \quad (110)$$

$$\nabla_{s'_M} = (\varepsilon_D \otimes M \otimes D) \circ (s'_M \otimes D) \circ (M \otimes (t'_{D,D} \circ \delta_D)). \quad (111)$$

Proof. Using Proposition 1.19, (c3) and the properties of the antipode and its inverse we get (96):

$$\begin{aligned} \nabla_{r_M} &= \\ &= (M \otimes \lambda_D) \circ \nabla_{r_M} \circ (M \otimes \lambda_D^{-1}) \\ &= (M \otimes (\lambda_D \circ \mu_D)) \circ (r'_M \otimes \lambda_D^{-1}) \circ (\eta_D \otimes M \otimes D) \\ &= (M \otimes (\mu_D \circ t_{D,D})) \circ (M \otimes \lambda_D \otimes D) \circ (r'_M \otimes D) \circ (\eta_D \otimes M \otimes D) \\ &= (M \otimes (\mu_D \circ t_{D,D})) \circ (r'_M \otimes D) \circ (\eta_D \otimes M \otimes D). \end{aligned}$$

In a similar way we obtain (97):

$$\begin{aligned} \nabla_{r_M} &= \\ &= (M \otimes \lambda_D^{-1}) \circ \nabla_{r_M} \circ (M \otimes \lambda_D) \\ &= (\varepsilon_D \otimes M \otimes D) \circ (r_M \otimes \lambda_D^{-1}) \circ (M \otimes (\delta_D \circ \lambda_D)) \\ &= (\varepsilon_D \otimes M \otimes D) \circ (r_M \otimes D) \circ (M \otimes \lambda_D \otimes D) \circ (M \otimes (t_{D,D} \circ \delta_D)) \\ &= (\varepsilon_D \otimes M \otimes D) \circ (r_M \otimes D) \circ (M \otimes (t_{D,D} \circ \delta_D)). \end{aligned}$$

The remaining equalities can be proved following the same pattern, composing with λ_D and λ_D^{-1} in the suitable order at convenience. \square

2. THE CATEGORY OF YETTER-DRINFELD MODULES

In this section the category of left-left Yetter-Drinfeld modules over an arbitrary WBHA D is defined. We deal with WBHA's in a monoidal category \mathcal{C} that is not assumed to be equipped with a braiding. In this situation, the first task consists on giving a suitable definition of Yetter-Drinfeld module such that we recovered the classic one in the particular case of modules over a Hopf algebra in a symmetric category as it appears in [20], and also the generalization of the preceding one to the weak Hopf algebra case introduced in [7].

In the definition of (M, D) -WO, we have only considered a WBHA D , while M was simply an arbitrary object of the monoidal category. It will be now discussed how the notion of (M, D) -WO can be enriched when the object M is also equipped with an algebraic structure.

Lemma 2.1. Let D be a WBHA, M in \mathcal{C} and (r_M, r'_M, s_M, s'_M) an (M, D) -WO. It holds that:

- (i) If (M, φ_M) is a left D -module then
 - (i-1) $\varphi_M = \varphi_M \circ \nabla_{s_M}$ iff $\varphi_M \circ s'_M \circ (M \otimes \eta_D) = id_M$,
 - (i-2) $\varphi_M = \varphi_M \circ \nabla_{r'_M}$ iff $\varphi_M \circ r_M \circ (M \otimes \eta_D) = id_M$.
- (ii) If (M, ϱ_M) is a left D -comodule then
 - (ii-1) $\varrho_M = \nabla_{s_M} \circ \varrho_M$ iff $(M \otimes \varepsilon_D) \circ s_M \circ \varrho_M = id_M$,
 - (ii-2) $\varrho_M = \nabla_{r'_M} \circ \varrho_M$ iff $(M \otimes \varepsilon_D) \circ r'_M \circ \varrho_M = id_M$.

Proof. For (i-1), to prove the direct implication, using the hypothesis, (c3) and the module condition, we have

$$\begin{aligned}
 & \varphi_M \\
 &= \varphi_M \circ (\mu_D \otimes M) \circ (D \otimes (s'_M \circ (M \otimes \eta_D))) \\
 &= \varphi_M \circ (D \otimes (\varphi_M \circ s'_M \circ (M \otimes \eta_D))),
 \end{aligned}$$

so composing with $\eta_D \otimes M$ the desired equality follows.

On the other hand, if $\varphi_M \circ s'_M \circ (M \otimes \eta_D) = id_M$ then

$$\begin{aligned}
 & \varphi_M \circ \nabla_{s_M} \\
 &= \varphi_M \circ (\mu_D \otimes M) \circ (D \otimes (s'_M \circ (M \otimes \eta_D))) \\
 &= \varphi_M \circ (D \otimes (\varphi_M \circ s'_M \circ (M \otimes \eta_D))) \\
 &= \varphi_M,
 \end{aligned}$$

and we obtain the opposite implication. The other statements follow similarly using (c3) at convenience. \square

Now we introduce the notion of weak operator compatible with a (co)module structure of M .

Definition 2.2. Let D be a WBHA, M an object of \mathcal{C} and (r_M, r'_M, s_M, s'_M) an (M, D) -WO.

- (i) If (M, φ_M) is a left D -module, the (M, D) -WO is said to be compatible with the D -module structure provided that it satisfies:
 - (i-1) $r_M \circ (\varphi_M \otimes D) = (D \otimes \varphi_M) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M)$,
 - (i-2) $r'_M \circ (D \otimes \varphi_M) = (\varphi_M \otimes D) \circ (D \otimes r'_M) \circ (t'_{D,D} \otimes M)$,
 - (i-3) $s'_M \circ (\varphi_M \otimes D) = (D \otimes \varphi_M) \circ (t'_{D,D} \otimes M) \circ (D \otimes s'_M)$,
 - (i-4) $s_M \circ (D \otimes \varphi_M) = (\varphi_M \otimes D) \circ (D \otimes s_M) \circ (t_{D,D} \otimes M)$.
- (ii) If (M, ϱ_M) is a left D -comodule, the (M, D) -WO is said to be compatible with the D -comodule structure provided that it satisfies:
 - (ii-1) $(D \otimes \varrho_M) \circ r_M = (t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (\varrho_M \otimes D)$,
 - (ii-2) $(\varrho_M \otimes D) \circ r'_M = (D \otimes r'_M) \circ (t'_{D,D} \otimes M) \circ (D \otimes \varrho_M)$,
 - (ii-3) $(D \otimes \varrho_M) \circ s'_M = (t'_{D,D} \otimes M) \circ (D \otimes s'_M) \circ (\varrho_M \otimes D)$,
 - (ii-4) $(\varrho_M \otimes D) \circ s_M = (D \otimes s_M) \circ (t_{D,D} \otimes M) \circ (D \otimes \varrho_M)$.

Notice that in the particular case of \mathcal{C} being a braided category with braiding c the conditions trivialize because of $t_{D,D} = c_{D,D}$, $t'_{D,D} = c_{D,D}^{-1}$, $r_M = c_{M,D}$, $r'_M = c_{M,D}^{-1}$, $s_M = c_{D,M}$ and $s'_M = c_{D,M}^{-1}$. Then in that context the compatibility is not a restriction.

Definition 2.3. Let D be a WBHA. We say that $(M, \varphi_M, \varrho_M)$ is a left-left Yetter-Drinfeld module over D if (M, φ_M) is a left D -module, (M, ϱ_M) is a left D -comodule and:

- (yd1) $\varrho_M = (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M) \circ (\eta_D \otimes M)$.
- (yd2) There exists (r_M, r'_M, s_M, s'_M) an (M, D) -WO compatible with the (co)module structure of M , such that

$$(\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)$$

$$= (\mu_D \otimes M) \circ (D \otimes r_M) \circ ((\varrho_M \circ \varphi_M) \otimes D) \circ (D \otimes s_M) \circ (\delta_D \otimes M).$$

The class of all left-left Yetter-Drinfeld modules over D will be denoted by ${}^D_D\mathcal{YD}$.

Remark 2.4. Note that when the ambient category \mathcal{C} is symmetric and we take both the weak Yang-Baxter operator and the (M, D) -WO to be the braiding of \mathcal{C} , we recover the classic definitions of Yetter-Drinfeld module introduced in [20] in the context of Hopf algebras and generalized in [8] (see also [9] and [16]) to the context of weak Hopf algebras.

Moreover, assuming that \mathcal{C} is braided with braiding c and $t_{D,D} = c_{D,D}$, $t'_{D,D} = c_{D,D}^{-1}$, if (M, φ_M) is a left D -module and (M, ϱ_M) a left D -comodule, $(c_{M,D}, c_{M,D}^{-1}, c_{D,M}, c_{D,M}^{-1})$ is an (M, D) -WO compatible with the (co)module structure of M . Therefore, we can define in this setting a left-left Yetter-Drinfeld module over D as a left D -module (M, φ_M) and a left D -comodule (M, ϱ_M) such that the following equalities hold:

- (i) $\varrho_M = (\mu_D \otimes \varphi_M) \circ (D \otimes c_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M) \circ (\eta_D \otimes M).$
- (ii) $(\mu_D \otimes \varphi_M) \circ (D \otimes c_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)$
 $= (\mu_D \otimes M) \circ (D \otimes c_{M,D}) \circ ((\varrho_M \circ \varphi_M) \otimes D) \circ (D \otimes c_{D,M}) \circ (\delta_D \otimes M).$

Definition 2.5. Let $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ be in the class ${}^D_D\mathcal{YD}$ with associated weak operators (r_M, r'_M, s_M, s'_M) and (r_N, r'_N, s_N, s'_N) respectively. It is said that a morphism $f : M \rightarrow N$ in \mathcal{C} is a morphism of left-left Yetter-Drinfeld modules if:

- (i) f is a left (co)module morphism.
- (ii) $r_N \circ (f \otimes D) = (D \otimes f) \circ r_M$, $s_N \circ (D \otimes f) = (f \otimes D) \circ s_M.$

Remark 2.6. In the last definition, the verification of the condition (ii) for r_M is equivalent to its verification for r'_M , and the same happens with s_M and s'_M . Actually, if we assume (ii) for r_M using the characterization of ∇_{r_M} of (c3-1) we conclude that:

$$\nabla_{r_N} \circ (f \otimes D) = (f \otimes D) \circ \nabla_{r_M}, \quad (112)$$

and by (c3-2) we deduce:

$$\nabla_{r'_N} \circ (D \otimes f) = (D \otimes f) \circ \nabla_{r'_M}. \quad (113)$$

Combining the preceding equalities with (c3) and part (ii) of Proposition 1.11 we conclude that $(f \otimes D) \circ r'_M = r'_N \circ (D \otimes f)$. Indeed,

$$\begin{aligned} & (f \otimes D) \circ r'_M \\ &= (f \otimes D) \circ \nabla_{r_M} \circ r'_M \\ &= \nabla_{r_N} \circ (f \otimes D) \circ r'_M \\ &= r'_N \circ r_N \circ (f \otimes D) \circ r'_M \\ &= r'_N \circ (D \otimes f) \circ \nabla_{r'_M} \\ &= r_N \circ \nabla_{r'_N} \circ (D \otimes f) \\ &= r'_N \circ (D \otimes f). \end{aligned}$$

The proof for the equality $s'_N \circ (f \otimes D) = (D \otimes f) \circ s'_M$ follows by the same argument.

As the identity morphism id_M satisfies the above conditions for any object M it can be introduced the following:

Definition 2.7. Let D be a WBHA. The category of left-left Yetter-Drinfeld modules is that whose objects are the class ${}^D_D\mathcal{YD}$ and whose morphisms between objects are those in the conditions of Definition 2.5. It will be denoted also by ${}^D_D\mathcal{YD}$.

Generalizing the braided symmetric case [[9], Proposition 2.2], the conditions (yd1) and (yd2) can also be restated in the following way:

Proposition 2.8. Let D be a WBHA and let (M, φ_M) be a left D -module and (M, ϱ_M) a left D -comodule. Assume that there exists (r_M, r'_M, s_M, s'_M) an (M, D) -WO compatible with the (co)module structures of M . Then the conditions (yd1) and (yd2) are equivalent to

$$\begin{aligned} \text{(yd3)} \quad & \varrho_M \circ \varphi_M \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D) \circ (D \otimes s_M) \\ & \quad \circ (\delta_D \otimes M). \end{aligned}$$

Proof. Indeed, if we assume (yd1) and (yd2) then:

$$\begin{aligned} & (\mu_D \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D) \circ (D \otimes s_M) \\ & \quad \circ (\delta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (\mu_D \otimes r_M) \circ (D \otimes r_M \otimes D) \circ ((\varrho_M \circ \varphi_M) \otimes D \otimes D) \circ (D \otimes s_M \otimes \lambda_D) \\ & \quad \circ (\delta_D \otimes s_M) \circ (\delta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ ((\varrho_M \circ \varphi_M) \otimes \Pi_D^L) \circ (D \otimes s_M) \circ (\delta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ ((\varrho_M \circ \varphi_M) \otimes D) \circ (\mu_D \otimes s_M) \circ (D \otimes t_{D,D} \otimes M) \\ & \quad \circ ((\delta_D \circ \eta_D) \otimes D \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ ((\varrho_M \circ \varphi_M) \otimes D) \circ (D \otimes s_M) \circ ((\delta_D \circ \eta_D) \otimes \varphi_M) \\ &= (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M) \circ (\eta_D \otimes \varphi_M) \\ &= \varrho_M \circ \varphi_M. \end{aligned}$$

In the preceding calculations, the first and fifth equalities follow by (yd2), the second by (c4) and the third one by (88) and (90). On the fourth equality we apply compatibility with the D -module structure and on the last one (yd1).

On the other hand, assuming (yd3) we can deduce (yd1) as follows:

$$\begin{aligned} & \varrho_M \\ &= \varrho_M \circ \varphi_M \circ (\eta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D) \circ (D \otimes s_M) \\ & \quad \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (D \otimes D \otimes \varrho_M)) \otimes D) \\ & \quad \circ (\delta_D \otimes s_M) \circ (D \otimes \Pi_D^R \otimes M) \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= ((\mu_D \circ (D \otimes (\overline{\Pi}_D^L \otimes \lambda_D))) \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \\ & \quad \circ (\delta_D \otimes \varrho_M)) \otimes D) \circ (D \otimes s_M) \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= (D \otimes (\varepsilon_D \circ \mu_D) \otimes M) \circ (\delta_D \otimes \lambda_D \otimes M) \circ (D \otimes r_M) \\ & \quad \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes D) \circ (D \otimes s_M) \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= (D \otimes (\varepsilon_D \circ \mu_D) \otimes M) \circ (\mu_D \otimes \mu_D \otimes D \otimes M) \circ (D \otimes t_{D,D} \otimes D \otimes r_M) \\ & \quad \circ (\delta_D \otimes \delta_D \otimes \varphi_M \otimes D) \circ (D \otimes t_{D,D} \otimes M \otimes D) \circ (D \otimes D \otimes \varrho_M \otimes \lambda_D) \circ (D \otimes D \otimes s_M) \\ & \quad \circ (D \otimes \delta_D \otimes M) \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= (D \otimes (\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes D \otimes r_M) \circ (\mu_D \otimes \mu_D \otimes \varphi_M \otimes D) \\ & \quad \circ (D \otimes t_{D,D} \otimes t_{D,D} \otimes M \otimes D) \circ (\delta_D \otimes t_{D,D} \otimes \varrho_M \otimes \lambda_D) \circ (D \otimes D \otimes D \otimes s_M) \\ & \quad \circ (D \otimes D \otimes t_{D,D} \otimes M) \circ (D \otimes \delta_D \otimes \varrho_M) \circ ((\delta_D \circ \eta_D) \otimes M) \\ &= (D \otimes (\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes D \otimes r_M) \circ (D \otimes \mu_D \otimes \varphi_M \otimes \lambda_D) \circ (D \otimes D \otimes t_{D,D} \otimes M \otimes D) \\ & \quad \circ (D \otimes \delta_D \otimes \varrho_M \otimes D) \circ (D \otimes D \otimes s_M) \circ (\mu_D \otimes \delta_D \otimes M) \circ (D \otimes t_{D,D} \otimes M) \\ & \quad \circ ((\delta_D \circ \eta_D) \otimes \varrho_M) \\ &= (D \otimes ((\varepsilon_D \otimes M) \circ \varrho_M)) \circ (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ ((\delta_D \circ \eta_D) \otimes \varrho_M) \\ &= (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ ((\delta_D \circ \eta_D) \otimes \varrho_M). \end{aligned}$$

The first equality follows by the condition of D -module for M . In the second and ninth ones we apply the hypothesis; the third one uses (94) and the equality

$$(D \otimes \lambda_D) \circ \delta_D \circ \eta_D = (D \otimes \Pi_D^R) \circ \delta_D \circ \eta_D.$$

The fourth equality relies on Proposition 1.18 and (30); the fifth is a consequence of the equality

$$\mu_D \circ (D \otimes \overline{\Pi}_D^L) = ((D \otimes (\varepsilon_D \circ \mu_D)) \circ (\delta_D \otimes D)),$$

and the sixth and eighth ones follow because of D is a WBHA. In the seventh equality we apply compatibility of the D -module structure for M ; finally, in the last one we use the condition of D -comodule for M .

Using the same technics we get:

$$\begin{aligned} & (\mu_D \otimes M) \circ (D \otimes r_M) \circ ((\varrho_M \circ \varphi_M) \otimes D) \circ (D \otimes s_M) \circ (\delta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (\mu_D \otimes r_M) \circ (D \otimes r_M \otimes D) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \\ & \quad \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D \otimes D) \circ (D \otimes s_M \otimes D) \circ (\delta_D \otimes s_M) \circ (\delta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \Pi_D^R) \circ (D \otimes s_M) \\ & \quad \circ (\delta_D \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes D) \circ (D \otimes s_M) \\ & \quad \circ (\mu_D \otimes \lambda_D \otimes M) \circ (D \otimes (\delta_D \circ \eta_D) \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (\mu_D \otimes \varphi_M \otimes D) \circ (D \otimes t_{D,D} \otimes M \otimes D) \circ (\mu_D \otimes \mu_D \otimes \varrho_M \otimes D) \\ & \quad \circ (D \otimes t_{D,D} \otimes D \otimes s_M) \circ (\delta_D \otimes \delta_D \otimes \lambda_D \otimes M) \circ (D \otimes (\delta_D \otimes \eta_D) \otimes M) \\ &= (\mu_D \otimes M) \circ (D \otimes r_M) \circ (\mu_D \otimes \varphi_M \otimes D) \circ (\mu_D \otimes t_{D,D} \otimes \varphi_M \otimes D) \\ & \quad \circ (D \otimes t_{D,D} \otimes t_{D,D} \otimes M \otimes D) \circ (\delta_D \otimes \delta_D \otimes \varrho_M \otimes \lambda_D) \circ (D \otimes D \otimes s_M) \\ & \quad \circ (D \otimes (\delta_D \otimes \eta_D) \otimes M) \\ &= (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes ((\mu_D \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \\ & \quad \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D) \circ (D \otimes s_M) \circ (\delta_D \otimes M))) \circ (D \otimes \eta_D \otimes M) \\ &= (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M) \circ (D \otimes (\varphi_M \circ (\eta_D \otimes M))) \\ &= (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M), \end{aligned}$$

so the condition (yd2) can be obtained from (yd3). \square

The following properties about Yetter-Drinfeld modules constitute a generalization of the results obtained in the braided context. See [5] for the idea of the proof.

Lemma 2.9. Let D be a WBHA in \mathcal{C} . If $(M, \varphi_M, \varrho_M)$ is in ${}^D_B\mathcal{YD}$ then it obeys the following properties:

$$\varrho_M \circ \varphi_M \circ (\Pi_D^L \otimes M) = (\mu_D \otimes D) \circ (\Pi_D^L \otimes \varrho_M), \quad (114)$$

$$(\Pi_D^L \otimes M) \circ \varrho_M \circ \varphi_M = (\Pi_D^L \otimes \varphi_M) \circ (\delta_D \otimes M), \quad (115)$$

$$\varrho_M \circ \varphi_M \circ (\Pi_D^R \otimes M) = (\mu_D \otimes M) \circ (D \otimes (\lambda_D \circ \Pi_D^R) \otimes M) \circ (D \otimes (r_M \circ s_M)) \circ (t_{D,D} \otimes M) \circ (D \otimes \varrho_M), \quad (116)$$

$$(\Pi_D^R \otimes M) \circ \varrho_M \circ \varphi_M = (D \otimes \varphi_M) \circ (t_{D,D} \otimes M) \circ (D \otimes (r_M \circ s_M)) \circ (D \otimes (\Pi_D^R \circ \lambda_D) \otimes M) \circ (\delta_D \otimes M). \quad (117)$$

Proposition 2.10. Let D be a WBHA with invertible antipode and let $(M, \varphi_M, \varrho_M)$ be in ${}^D_D\mathcal{YD}$. Then:

- (i) $\varphi_M = \varphi_M \circ \nabla_{s_M}$ iff $\varrho_M = \nabla_{s_M} \circ \varrho_M$,
- (ii) $\varphi_M = \varphi_M \circ \nabla_{r'_M}$ iff $\varrho_M = \nabla_{r'_M} \circ \varrho_M$.

Proof. We will show (i), being (ii) analogous. For the ‘if’ part, in virtue of Proposition 1.18, the equality $(\bar{\Pi}_D^L \otimes M) \circ \varrho_M = (D \otimes \varphi_M) \circ ((\delta_D \circ \eta_D) \otimes M)$ which holds by (115), compatibility of the module structure, Corollary 1.20 and the hypothesis, it results that

$$\begin{aligned}
& (M \otimes \varepsilon_D) \circ s_M \circ \varrho_M \\
&= (M \otimes \varepsilon_D) \circ s_M \circ (\bar{\Pi}_D^L \otimes M) \circ \varrho_M \\
&= (M \otimes \varepsilon_D) \circ s_M \circ (D \otimes \varphi_M) \circ ((\delta_D \circ \eta_D) \otimes M) \\
&= (\varphi_M \otimes \varepsilon_D) \circ (D \otimes s_M) \circ (t_{D,D} \otimes M) \circ ((\delta_D \circ \eta_D) \otimes M) \\
&= \varphi_M \circ \nabla_{s_M} \circ (\eta_D \otimes M) \\
&= \varphi_M \circ (\eta_D \otimes M) \\
&= id_M.
\end{aligned}$$

Applying Lemma 2.1 we obtain the equality $\varrho_M = \nabla_{s_M} \circ \varrho_M$.

The opposite implication follows a similar pattern:

$$\begin{aligned}
& \varphi_M \circ s'_M \circ (M \otimes \eta_D) \\
&= \varphi_M \circ (\Pi_D^L \otimes M) \circ s'_M \circ (M \otimes \eta_D) \\
&= ((\varepsilon_D \circ \mu_D) \otimes M) \circ (\Pi_D^L \otimes \varrho_M) \circ s'_M \circ (M \otimes \eta_D) \\
&= ((\varepsilon_D \circ \mu_D \circ t'_{D,D}) \otimes M) \circ (D \otimes s'_M) \circ (\varrho_M \otimes \eta_D) \\
&= (\varepsilon_D \otimes M) \circ \nabla_{s_M} \circ \varrho_M \\
&= (\varepsilon_D \otimes M) \circ \varrho_M \\
&= id_M,
\end{aligned}$$

and by Lemma 2.1 we have that $\varphi_M = \varphi_M \circ \nabla_{s_M}$. □

Corollary 2.11. Let D be a WBHA with invertible antipode and $(M, \varphi_M, \varrho_M)$ an object in ${}^D_D\mathcal{YD}$. It holds that

$$\varphi_M \circ \nabla_{s_M} = \varphi_M \circ \nabla_{r'_M} = \varphi_M; \quad \nabla_{s_M} \circ \varrho_M = \nabla_{r'_M} \circ \varrho_M = \varrho_M. \quad (118)$$

Proof. Using that M is a D -comodule, the condition (yd-3) twice, (c4) and the counit property we can write

$$\begin{aligned}
& \varphi_M \circ \nabla_{s_M} \\
&= (\varepsilon_D \otimes M) \circ \varrho_M \circ \varphi_M \circ \nabla_{s_M} \\
&= ((\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D) \\
&\quad \circ (D \otimes s_M \otimes \varepsilon_D) \circ (\delta_D \otimes s_M) \circ (\delta_D \otimes M) \\
&= ((\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D) \\
&\quad \circ (D \otimes M \otimes ((D \otimes \varepsilon_D) \circ \delta_D)) \circ (D \otimes s_M) \circ (\delta_D \otimes M) \\
&= ((\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes r_M) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes \lambda_D) \\
&\quad \circ (D \otimes s_M) \circ (\delta_D \otimes M) \\
&= (\varepsilon_D \otimes M) \circ \varrho_M \circ \varphi_M \\
&= \varphi_M.
\end{aligned}$$

Now, by Proposition 2.10 we also know that $\varrho_M = \nabla_{s_M} \circ \varrho_M$. The remaining equalities can be proved by similar arguments. □

In this part of the work the announced monoidal structure of ${}^D_D\mathcal{YD}$ in the general case is presented. We want also to point out that when we restrict to the braided case we recover the monoidal structure exposed in [5], so it could be said that the new theory introduced in this work is coherent with the classic one developed in the Hopf algebra setting.

2.12. Let D be a WBHA. If (M, φ_M) and (N, φ_N) are left D -modules and it exists a quadruple (r_M, r'_M, s_M, s'_M) forming an (M, D) -WO compatible with the (co)module structure, then two different morphisms arise naturally:

$$\begin{aligned}\nabla_{M \otimes N} &= (\varphi_M \otimes \varphi_N) \circ (D \otimes s_M \otimes N) \circ ((\delta_D \circ \eta_D) \otimes M \otimes N), \\ \Delta_{M \otimes N} &= ((\varepsilon_D \circ \mu_D) \otimes M \otimes N) \circ (D \otimes r_M \otimes N) \circ (\varrho_M \otimes \varrho_N).\end{aligned}$$

Lemma 2.13. Let D be a WBHA. If (M, φ_M) and (N, φ_N) are left D -modules and (r_M, r'_M, s_M, s'_M) is an (M, D) -WO compatible with the module structure, then the morphism $\nabla_{M \otimes N}$ is idempotent. It holds the analogous result for $\Delta_{M \otimes N}$ in the comodule case.

Proof. We will give the proof for $\nabla_{M \otimes N}$. Using the compatibility, the module character of M and N and the conditions (c4) and (b4) we have:

$$\begin{aligned}& \nabla_{M \otimes N} \circ \nabla_{M \otimes N} \\ &= (\varphi_M \otimes \varphi_N) \circ (D \otimes \varphi_M \otimes D \otimes \varphi_N) \circ (D \otimes D \otimes s_M \otimes D \otimes N) \circ (D \otimes t_{D,D} \otimes s_M \otimes N) \\ & \quad \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D) \otimes M \otimes N) \\ &= (\varphi_M \otimes \varphi_N) \circ (D \otimes s_M \otimes N) \circ (((\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D)) \\ & \quad \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))) \otimes M \otimes N \\ &= \nabla_{M \otimes N}.\end{aligned}$$

□

The following two lemmas have been introduced as technical tools to be used in order to show that the morphisms $\nabla_{M \otimes N}$ and $\Delta_{M \otimes N}$ coincide.

Lemma 2.14. Let D be a WBHA with invertible antipode. If $(M, \varphi_M, \varrho_M)$ is a left D -(co)module and (r_M, s'_M, s_M, s'_M) an (M, D) -WO compatible with the (co)module structure, then

$$\nabla_{s'_M} \circ \nabla_{r_M} = \nabla_{r_M} \circ \nabla_{s'_M}. \quad (119)$$

Proof. Using the properties of WBHA, (98) twice, (71) and (55), we have:

$$\begin{aligned}& \nabla_{s'_M} \circ \nabla_{r_M} \\ &= (\varepsilon_D \otimes \nabla_{s'_M}) \circ (r_M \otimes D) \circ (M \otimes (t_{D,D} \circ \delta_D)) \\ &= (\varepsilon_D \otimes M \otimes D) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (\nabla_{s'_M} \otimes D) \circ (M \otimes \delta_D) \\ &= (\varepsilon_D \otimes M \otimes D) \circ (r_M \otimes D) \circ (M \otimes (t_{D,D} \circ \delta_D)) \circ \nabla_{s'_M} \\ &= \nabla_{r_M} \circ \nabla_{s'_M}.\end{aligned}$$

□

As a consequence:

Lemma 2.15. In the hypothesis of the previous lemma, if it also holds that $(M, \varphi_M, \varrho_M) \in {}^D_D\mathcal{YD}$ then

$$(M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{s'_M} \circ r'_M \circ \varrho_M) \otimes D) = (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((r'_M \circ \varrho_M) \otimes D). \quad (120)$$

Proof. Indeed,

$$(M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{s'_M} \circ r'_M \circ \varrho_M) \otimes D)$$

$$\begin{aligned}
&= (M \otimes (\varepsilon_D \circ \mu_D \circ (\overline{\Pi}_D^R \otimes D))) \circ ((\nabla_{s'_M} \circ r'_M \circ \varrho_M) \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{s'_M} \circ r'_M \circ (\overline{\Pi}_D^R \otimes M) \circ \varrho_M) \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{s'_M} \circ r'_M) \otimes D) \circ (D \otimes \varphi_M \otimes D) \circ (t_{D,D} \otimes M \otimes D) \\
&\quad \circ (D \otimes (r_M \circ s_M) \otimes D) \circ ((\delta_D \circ \eta_D) \otimes M \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{s'_M} \circ r_M) \otimes D) \circ (\varphi_M \otimes D \otimes D) \circ (D \otimes s_M \otimes D) \\
&\quad \circ ((\delta_D \circ \eta_D) \otimes M \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{r_M} \circ \nabla_{s'_M} \circ (\varphi_M \otimes D) \circ (D \otimes s_M) \\
&\quad \circ ((t_{D,D} \circ t'_{D,D} \circ \delta_D \circ \eta_D) \otimes M)) \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ (\nabla_{r_M} \otimes D) \circ (s_M \otimes D) \circ (D \otimes \varphi_M \otimes D) \circ ((t'_{D,D} \circ \delta_D \circ \eta_D) \otimes M \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{r_M} \circ (\varphi_M \otimes D) \circ (D \otimes s_M) \circ ((t_{D,D} \circ t'_{D,D} \circ \delta_D \circ \eta_D) \otimes M)) \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((\nabla_{r_M} \circ (\varphi_M \otimes D) \circ (D \otimes s_M) \circ ((\delta_D \circ \eta_D) \otimes M)) \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ (r'_M \otimes D) \circ ((\overline{\Pi}_D^R \otimes M) \circ \varrho_M) \otimes D) \\
&= (M \otimes (\varepsilon_D \circ \mu_D)) \circ ((r'_M \circ \varrho_M) \otimes D).
\end{aligned}$$

In the preceding calculations, the first and the last equalities follow because $\varepsilon_D \circ \mu_D \circ (\overline{\Pi}_D^R \otimes D) = \varepsilon_D \circ \mu_D$. The second uses Propositions 1.17 and 1.18. We get the third and the tenth ones by the equation

$$(\overline{\Pi}_D^R \otimes M) \circ \varrho_M = (D \otimes \varphi_M) \circ (t_{D,D} \otimes M) \circ ((D \otimes r_M \circ s_M)) \circ ((\delta_D \circ \eta_D) \otimes M), \quad (121)$$

which follows by Lemma 2.9. In the fourth and seventh equalities we apply the compatibility condition for the module structure; the fifth relies on Lemma 2.14, and the sixth is a consequence of (60). Finally, the ninth equality follows by (b2-1). \square

Now it is possible to check that the idempotent morphisms defined in paragraph 2.12 are the same.

Proposition 2.16. Let D be a WBHA with invertible antipode. If $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ are objects of ${}^D_B\mathcal{YD}$ then

$$\nabla_{M \otimes N} = \Delta_{M \otimes N}.$$

Proof. We have:

$$\begin{aligned}
&\nabla_{M \otimes N} \\
&= ((\varphi_M \circ s'_M) \otimes \varphi_N) \circ (M \otimes (\delta_D \circ \eta_D) \otimes N) \\
&= ((\varphi_M \circ s'_M) \otimes N) \circ (M \otimes ((\overline{\Pi}_D^L \otimes N) \circ \varrho_N)) \\
&= (((\varepsilon_D \circ \mu_D) \otimes M) \circ (D \otimes (r_M \circ s_M)) \circ (t_{D,D} \otimes M) \circ (D \otimes \varrho_M)) \otimes N \circ (s'_M \otimes N) \\
&\quad \circ (M \otimes \varrho_N) \\
&= (((M \otimes (\varepsilon_D \circ \mu_D)) \circ (r'_M \otimes D) \circ (D \otimes s_M) \circ ((t_{D,D} \circ t'_{D,D}) \otimes M) \circ (D \otimes s'_M)) \otimes N) \\
&\quad \circ (\varrho_M \otimes \varrho_N) \\
&= (((M \otimes (\varepsilon_D \circ \mu_D \circ t_{D,D})) \circ (s_M \otimes D) \circ (D \otimes r'_M) \circ (t'_{D,D} \otimes M) \circ (D \otimes s'_M)) \otimes N) \\
&\quad \circ (\varrho_M \otimes \varrho_N) \\
&= (((M \otimes (\varepsilon_D \circ \mu_D \circ t_{D,D})) \circ (\nabla_{s'_M} \otimes D) \circ (M \otimes t'_{D,D}) \circ (r'_M \otimes D)) \otimes N) \circ (\varrho_M \otimes \varrho_N) \\
&= (((M \otimes (\varepsilon_D \circ \mu_D \circ t_{D,D})) \circ (M \otimes \mu_D \otimes D) \circ ((s_M \otimes (\eta_D \otimes M)) \otimes t'_{D,D}) \\
&\quad \circ (r'_M \otimes D)) \otimes N) \circ (\varrho_M \otimes \varrho_N) \\
&= (((M \otimes (\varepsilon_D \circ \mu_D)) \circ (M \otimes (\mu_D \circ t_{D,D}) \otimes D) \circ ((s_M \otimes (\eta_D \otimes M)) \otimes \nabla_{D,D}) \\
&\quad \circ (r'_M \otimes D)) \otimes N) \circ (\varrho_M \otimes \varrho_N)
\end{aligned}$$

$$\begin{aligned}
&= (M \otimes (\varepsilon_D \circ \mu_D) \otimes N) \circ ((\nabla_{s'_M} \circ r'_M \circ \varrho_M) \otimes \varrho_N) \\
&= (M \otimes (\varepsilon_D \circ \mu_D) \otimes N) \circ (r'_M \otimes D \otimes N) \circ (\varrho_M \otimes \varrho_N) \\
&= \Delta_{M \otimes N}.
\end{aligned}$$

In the preceding calculations, the first and the last equalities follow by (80) and (79), respectively; the second and third ones are consequences of Lemma 2.9; the fourth apply compatibility of the comodule structure. In the fifth and seventh equalities we use (c2-1) and (c3-2), respectively; the sixth follows by (77), and the eighth and ninth follow because D is a WBHA. Finally, the tenth equality relies on (120). \square

2.17. Let D be a WBHA and let $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ be objects of ${}^D_D\mathcal{YD}$. We denote by $M \times N$ the image of the idempotent $\nabla_{M \otimes N}$ and by $p_{M \otimes N} : M \otimes N \rightarrow M \times N$, $i_{M \otimes N} : M \times N \rightarrow M \otimes N$ the morphisms such that $i_{M \otimes N} \circ p_{M \otimes N} = \nabla_{M \otimes N}$ and $p_{M \otimes N} \circ i_{M \otimes N} = id_{M \times N}$. Actually the object $M \times N$ will be taken as the product of M and N in the category ${}^D_D\mathcal{YD}$. In order to provide ${}^D_D\mathcal{YD}$ with a monoidal structure, first to all, by Definition 2.3, the object $M \times N$ must be equipped with a compatible a weak operator. To do so, we state first some preliminary results and convenient notation.

Lemma 2.18. Let D be a WBHA with invertible antipode. If $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ are in ${}^D_D\mathcal{YD}$ then:

$$(D \otimes \nabla_{M \otimes N}) \circ (r_M \otimes N) \circ (M \otimes r_N) = (r_M \otimes N) \circ (M \otimes r_N) \circ (\nabla_{M \otimes N} \otimes D), \quad (122)$$

$$(\nabla_{M \otimes N} \otimes D) \circ (M \otimes r'_N) \circ (r'_M \otimes N) = (M \otimes r'_N) \circ (r'_M \otimes N) \circ (D \otimes \nabla_{M \otimes N}), \quad (123)$$

$$(\nabla_{M \otimes N} \otimes D) \circ (M \otimes s_N) \circ (s_M \otimes N) = (M \otimes s_N) \circ (s_M \otimes N) \circ (D \otimes \nabla_{M \otimes N}), \quad (124)$$

$$(D \otimes \nabla_{M \otimes N}) \circ (s'_M \otimes N) \circ (M \otimes s'_N) = (s'_M \otimes N) \circ (M \otimes s'_N) \circ (\nabla_{M \otimes N} \otimes D). \quad (125)$$

Proof. We will show (122), the others being analogous. First at all, using (yd-1) twice, the compatibility with the module structure, (b1) and (a2-4),

$$\begin{aligned}
&(\nabla_{D,D} \otimes N) \circ (D \otimes r_N) \circ (\varrho_N \otimes D) \\
&= (\nabla_{D,D} \otimes N) \circ (D \otimes r_N) \circ (((\mu_D \otimes \varphi_N) \circ (D \otimes t_{D,D} \otimes N) \circ ((\delta_D \circ \eta_D) \otimes \varrho_N)) \otimes D) \\
&= (\nabla_{D,D} \otimes \varphi_N) \circ (\mu_D \otimes t_{D,D} \otimes N) \circ (D \otimes t_{D,D} \otimes r_N) \circ ((\delta_D \circ \eta_D) \otimes \varrho_N \otimes D) \\
&= (\mu_D \otimes D \otimes \varphi_N) \circ (D \otimes \nabla_{D,D} \otimes D \otimes N) \circ (D \otimes D \otimes t_{D \otimes D} \otimes N) \circ (D \otimes t_{D,D} \otimes r_N) \\
&\quad \circ ((\delta_D \circ \eta_D) \otimes \varrho_N \otimes D) \\
&= (\mu_D \otimes D \otimes \varphi_N) \circ (D \otimes D \otimes t_{D,D} \otimes N) \circ (D \otimes (\nabla_{D,D} \circ t_{D,D}) \otimes r_N) \\
&\quad \circ ((\delta_D \circ \eta_D) \otimes \varrho_N \otimes D) \\
&= (D \otimes r_N) \circ (\varrho_N \otimes D).
\end{aligned}$$

Now, by the characterization $\nabla_{M \otimes N} = \Delta_{M \otimes N}$ obtained in Proposition 2.16, the compatibilities with the comodule structures, the conditions (c1) and (b3) and the equalities (19) and (34) we get:

$$\begin{aligned}
&(D \otimes \nabla_{M \otimes N}) \circ (r_M \otimes N) \circ (M \otimes r_N) \\
&= (D \otimes (((\varepsilon_D \circ \mu_D) \otimes M \otimes N) \circ (D \otimes r_M \otimes N) \circ (\varrho_M \otimes \varrho_N))) \circ (r_M \otimes N) \circ (M \otimes r_N)
\end{aligned}$$

$$\begin{aligned}
&= (((D \otimes (\varepsilon_D \circ \mu_D) \otimes M) \circ (t_{D,D} \otimes r_M) \circ (D \otimes r_M \otimes D) \circ (D \otimes M \otimes t_{D,D})) \otimes N) \\
&\quad \circ (D \otimes M \otimes D \otimes r_N) \circ (\varrho_M \otimes \varrho_N \otimes D) \\
&= (((D \otimes (\varepsilon_D \circ \mu_D) \otimes M) \circ (t_{D,D} \otimes D \otimes M) \circ (D \otimes t_{D,D} \otimes M) \circ (D \otimes D \otimes r_M)) \otimes N) \\
&\quad \circ (D \otimes r_M \otimes r_N) \circ (\varrho_M \otimes \varrho_N \otimes D) \\
&= (((D \otimes \varepsilon_D \otimes M) \circ (t_{D,D} \otimes M) \circ (\mu_D \otimes r_M)) \otimes N) \circ (D \otimes r_M \otimes r_N) \\
&\quad \circ (\varrho_M \otimes \varrho_N \otimes D) \\
&= (((\varepsilon_D \circ \mu_D) \otimes D \otimes M) \circ (D \otimes \nabla_{D,D} \otimes M) \circ (D \otimes D \otimes r_M)) \otimes N \circ (D \otimes r_M \otimes r_N) \\
&\quad \circ (\varrho_M \otimes \varrho_N \otimes D) \\
&= ((\varepsilon_D \circ \mu_D) \otimes r_M \otimes N) \circ (D \otimes r_M \otimes D \otimes N) \circ (\varrho_M \otimes \nabla_{D,D} \otimes N) \circ (M \otimes D \otimes r_N) \\
&\quad \circ (M \otimes \varrho_N \otimes D) \\
&= (r_M \otimes N) \circ (M \otimes r_N) \circ (\nabla_{M \otimes N} \otimes D).
\end{aligned}$$

□

2.19. Let D be a WBHA with invertible antipode. Given $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ in ${}^D_D\mathcal{YD}$ we denote by $\varphi_{M \otimes N}$ the morphism: $\varphi_{M \otimes N} : D \otimes M \otimes N \rightarrow M \otimes N$ defined by

$$\varphi_{M \otimes N} = (\varphi_M \otimes \varphi_N) \circ (D \otimes s_M \otimes N) \circ (\delta_D \otimes M \otimes N),$$

and by $\varrho_{M \otimes N}$ the morphism $\varrho_{M \otimes N} : M \otimes N \rightarrow D \otimes M \otimes N$ defined by

$$\varrho_{M \otimes N} = (\mu_D \otimes M \otimes N) \circ (D \otimes r_M \otimes N) \circ (\varrho_M \otimes \varrho_N).$$

Note that $\nabla_{M \otimes N} = \varphi_{M \otimes N} \circ (\eta_D \otimes M \otimes N) = (\varepsilon_D \otimes M \otimes N) \circ \varrho_{M \otimes N}$.

Using this notation and the compatibility with the correspondent weak operators, it results that:

$$\varphi_{M \otimes N} \circ (D \otimes \nabla_{M \otimes N}) = \varphi_{M \otimes N} = \nabla_{M \otimes N} \circ \varphi_{M \otimes N}, \quad (126)$$

$$\varrho_{M \otimes N} \circ \nabla_{M \otimes N} = \varrho_{M \otimes N} = (D \otimes \nabla_{M \otimes N}) \circ \varrho_{M \otimes N}. \quad (127)$$

Moreover, being $(P, \varphi_P, \varrho_P)$ in ${}^D_D\mathcal{YD}$ and combining the above equalities with Lemma 2.18 we obtain

$$(i_{M \otimes N} \otimes P) \circ \nabla_{(M \times N) \otimes P} \circ (p_{M \otimes N} \otimes P) = (M \otimes i_{N \otimes P}) \circ \nabla_{M \otimes (N \times P)} \circ (M \otimes p_{N \otimes P}), \quad (128)$$

and

$$\begin{aligned}
(M \otimes i_{N \otimes P}) \circ \nabla_{M \otimes (N \times P)} \circ (M \otimes p_{N \otimes P}) &= (\nabla_{M \otimes N} \otimes P) \circ (M \otimes \nabla_{N \otimes P}) = \\
&= (M \otimes \nabla_{N \otimes P}) \circ (\nabla_{M \otimes N} \otimes P).
\end{aligned} \quad (129)$$

Proposition 2.20. Let D be a WBHA with invertible antipode in \mathcal{C} . Given $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ in ${}^D_D\mathcal{YD}$, the quadruple $(r_{M \times N}, r'_{M \times N}, s_{M \times N}, s'_{M \times N})$ is an $(M \times N, D)$ -WO, where:

$$\begin{aligned}
r_{M \times N} &= (D \otimes p_{M \otimes N}) \circ (r_M \otimes N) \circ (M \otimes r_N) \circ (i_{M \otimes N} \otimes D), \\
r'_{M \times N} &= (p_{M \otimes N} \otimes D) \circ (M \otimes r'_N) \circ (r'_M \otimes N) \circ (D \otimes i_{M \otimes N}), \\
s_{M \times N} &= (p_{M \otimes N} \otimes D) \circ (M \otimes s_N) \circ (s_M \otimes N) \circ (D \otimes i_{M \otimes N}), \\
s'_{M \times N} &= (D \otimes p_{M \otimes N}) \circ (s'_M \otimes N) \circ (M \otimes s'_N) \circ (i_{M \otimes N} \otimes D).
\end{aligned}$$

Proof. We must check that the conditions stated in Definition 1.9 are satisfied. The proof of (c1), (c2) and (c4) consists basically on use twice these conditions referred to M and N , apply the statements obtained in Lemma 2.18 and the equality $\nabla_{M \otimes N} = i_{M \otimes N} \circ p_{M \otimes N}$. We write (c1-1) to illustrate the procedure:

$$\begin{aligned}
&(D \otimes r_{M \times N}) \circ (r_{M \times N} \otimes D) \circ (M \times N \otimes t_{D,D}) \\
&= (D \otimes D \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes N) \circ (D \otimes M \otimes r_N) \circ (((D \otimes \nabla_{M \otimes N}) \circ (r_M \otimes N)
\end{aligned}$$

$$\begin{aligned}
& \circ (M \otimes r_N) \otimes D) \circ (i_{M \otimes N} \otimes t_{D,D}) \\
&= (D \otimes D \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes N) \circ (D \otimes M \otimes r_N) \circ (r_M \otimes N \otimes D) \circ (M \otimes r_N \otimes D) \\
& \quad \circ (i_{M \otimes N} \otimes t_{D,D}) \\
&= (t_{D,D} \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes N) \circ (r_M \otimes r_N) \circ (M \otimes r_N \otimes D) \circ (i_{M \otimes N} \otimes D \otimes D) \\
&= (t_{D,D} \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes N) \circ (D \otimes M \otimes r_N) \circ (((D \otimes \nabla_{M \otimes N}) \circ (r_M \otimes N) \\
& \quad \circ (M \otimes r_N) \circ (i_{M \otimes N} \otimes D)) \otimes D) \\
&= (t_{D,D} \otimes M \times N) \circ (D \otimes r_{M \times N}) \circ (r_{M \times N} \otimes D).
\end{aligned}$$

The condition (c5) follows directly applying Proposition 1.19 twice for M and N .

As far as the condition (c3), we prove only (c3-1) because the others are analogous. Using the definition of $\nabla_{M \times N}$, Lemma 2.18, the condition (c3-1) referred to M and N , and the condition (c4) referred to N it follows that:

$$\begin{aligned}
& \nabla_{r_{M \times N}} \\
&= r'_{M \times N} \circ r_{M \times N} \\
&= (p_{M \otimes N} \otimes D) \circ (M \otimes r'_N) \circ (r'_M \otimes N) \circ (D \otimes \nabla_{M \otimes N}) \circ (r_M \otimes N) \circ (M \otimes r_N) \\
& \quad \circ (i_{M \otimes N} \otimes D) \\
&= (p_{M \otimes N} \otimes D) \circ (M \otimes r'_N) \circ (\nabla_{r_M} \otimes N) \circ (M \otimes r_N) \circ (i_{M \otimes N} \otimes D) \\
&= (\varepsilon_D \otimes p_{M \otimes N} \otimes D) \circ (r_M \otimes r'_N) \circ (M \otimes \delta_D \otimes N) \circ (M \otimes r_N) \circ (i_{M \otimes N} \otimes D) \\
&= (\varepsilon_D \otimes p_{M \otimes N} \otimes D) \circ (r_M \otimes \nabla_{r_N}) \circ (M \otimes r_N \otimes D) \circ (i_{M \otimes N} \otimes \delta_D) \\
&= (((\varepsilon_D \otimes p_{M \otimes N}) \circ (r_M \otimes \varepsilon_D \otimes N) \circ (M \otimes D \otimes r_N)) \otimes D) \circ (M \otimes r_N \otimes \delta_D) \circ (i_{M \otimes N} \otimes \delta_D) \\
&= (((\varepsilon_D \otimes p_{M \otimes N}) \circ (r_M \otimes \varepsilon_D \otimes N) \circ (M \otimes \delta_D \otimes N) \circ (M \otimes r_N)) \otimes D) \circ (i_{M \otimes N} \otimes \delta_D) \\
&= (((\varepsilon_D \otimes M \times N) \circ r_{M \times N}) \otimes D) \circ (M \times N \otimes \delta_D).
\end{aligned}$$

□

Proposition 2.21. Let D be a WBHA with invertible antipode. If $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ are objects in ${}^D_D\mathcal{YD}$, then $(M \times N, \varphi_{M \times N}, \varrho_{M \times N})$ is in ${}^D_D\mathcal{YD}$, where

$$\varphi_{M \times N} = p_{M \otimes N} \circ \varphi_{M \otimes N} \circ (D \otimes i_{M \otimes N}), \quad (130)$$

$$\varrho_{M \times N} = (D \otimes p_{M \otimes N}) \circ \varrho_{M \otimes N} \circ i_{M \otimes N}. \quad (131)$$

Proof. In Proposition 2.20, an $(M \times N, D)$ -WO is explicitly defined, so it only remains to prove that $\varphi_{M \times N}$ and $\varrho_{M \times N}$ are compatible (co)module structures satisfying the conditions (yd1) and (yd2). We leave to the reader to show that $(M \times N, \varphi_{M \times N})$ is a left D -module and $(M \times N, \varrho_{M \times N})$ is a left D -comodule. As far as compatibility, using compatibilities for M and N , the condition (b3-3) and the equalities (126) and (76) referred to M we have:

$$\begin{aligned}
& r_{M \times N} \circ (\varphi_{M \times N} \otimes D) \\
&= (D \otimes p_{M \otimes N}) \circ (r_M \otimes N) \circ (M \otimes r_N) \circ ((\nabla_{M \otimes N} \circ \varphi_{M \otimes N} \circ (D \otimes i_{M \otimes N})) \otimes D) \\
&= (D \otimes p_{M \otimes N}) \circ (r_M \otimes N) \circ (M \otimes r_N) \circ (((\varphi_M \otimes \varphi_N) \circ (D \otimes s_M \otimes N) \\
& \quad \circ (\delta_D \otimes i_{M \otimes N})) \otimes D) \\
&= (D \otimes p_{M \otimes N}) \circ (D \otimes \varphi_M \otimes N) \circ (t_{D,D} \otimes M \otimes N) \circ (D \otimes ((r_M \otimes \varphi_N) \circ (M \otimes t_{D,D} \otimes N) \\
& \quad \circ (s_M \otimes r_N))) \circ (\delta_D \otimes i_{M \otimes N} \otimes D) \\
&= (D \otimes (p_{M \otimes N} \circ (\varphi_M \otimes \varphi_N))) \circ (t_{D,D} \otimes s_M \otimes N) \circ (D \otimes t_{D,D} \otimes M \otimes N) \\
& \quad \circ (\delta_D \otimes ((r_M \otimes N) \circ (M \otimes r_N) \circ (i_{M \otimes N} \otimes D))) \\
&= (D \otimes p_{M \otimes N}) \circ (D \otimes (\varphi_{M \otimes N} \circ (D \otimes \nabla_{M \otimes N}))) \circ (t_{D,D} \otimes M \otimes N) \\
& \quad \circ (D \otimes ((r_M \otimes N) \circ (M \otimes r_N) \circ (i_{M \otimes N} \otimes D))) \\
&= (D \otimes \varphi_{M \times N}) \circ (t_{D,D} \otimes M \times N) \circ (D \otimes r_{M \times N}).
\end{aligned}$$

The proofs for $r'_{M \times N}$, $s_{M \times N}$ and $s'_{M \times N}$, are analogous. By similar arguments we get the result for the comodule structure.

To prove the condition (yd1) we write:

$$\begin{aligned}
& \varphi_{M \times N} \circ (\Pi_D^L \otimes M \times N) \circ \varrho_{M \times N} \\
&= p_{M \otimes N} \circ (\varphi_M \otimes \varphi_N) \circ (D \otimes s_M \otimes N) \circ ((\delta_D \circ \Pi_D^L) \otimes M \otimes N) \circ \varrho_{M \otimes N} \circ i_{M \otimes N} \\
&= p_{M \otimes N} \circ (\varphi_M \otimes \varphi_N) \circ (\mu_D \otimes s_M \otimes N) \circ (\Pi_D^L \otimes (\delta_D \circ \eta_D) \otimes M \otimes N) \circ \varrho_{M \otimes N} \circ i_{M \otimes N} \\
&= p_{M \otimes N} \circ (\varphi_M \otimes N) \circ (\Pi_D^L \otimes (((\varphi_M \circ s'_M) \otimes \varphi_N) \circ (M \otimes (\delta_D \circ \eta_D) \otimes N))) \circ \varrho_{M \otimes N} \circ i_{M \otimes N} \\
&= p_{M \otimes N} \circ (\varphi_M \otimes N) \circ (\Pi_D^L \otimes \nabla_{M \otimes N}) \circ \varrho_{M \otimes N} \circ i_{M \otimes N} \\
&= p_{M \otimes N} \circ (\varphi_M \otimes N) \circ (\Pi_D^L \otimes i_{M \otimes N}) \circ \varrho_{M \times N} \\
&= ((\varepsilon_D \circ \mu_D) \otimes M \times N) \circ (\Pi_D^L \otimes \varrho_{M \times N}) \circ \varrho_{M \times N} \\
&= ((\varepsilon_D \circ (\Pi_D^L \wedge id_D)) \otimes M \times N) \circ \varrho_{M \times N} \\
&= id_{M \times N}.
\end{aligned}$$

In the preceding calculations, the first equality follows by (127), while in the second one we apply that $\delta_D \circ \Pi_D^L = (\mu_D \otimes D) \circ (\Pi_D^L \otimes (\delta_D \circ \eta_D))$. In the third one we use (80); the fourth one follows because of the characterization

$$\nabla_{M \otimes N} = ((\varphi_M \circ s'_M) \otimes \varphi_N) \circ (M \otimes (\delta_D \circ \eta_D) \otimes N)$$

obtained in the proof of Proposition 2.16. The fifth equality follows by the definition of $\varrho_{M \times N}$, and the sixth one by the equality

$$p_{M \otimes N} \circ (\varphi_M \otimes N) \circ (\Pi_D^L \otimes i_{M \otimes N}) = ((\varepsilon_D \circ \mu_D) \otimes M \times N) \circ (\Pi_D^L \otimes \varrho_{M \times N}),$$

that in turn can be deduced using (b5) and (114). Finally, in the seventh equality we use that $(M \times N, \varrho_{M \times N})$ is a left D comodule and the last one follows by (26).

As a consequence (yd1) holds:

$$\begin{aligned}
& (\mu_D \otimes \varphi_{M \times N}) \circ (D \otimes t_{D,D} \otimes M \times N) \circ ((\delta_D \circ \eta_D) \otimes \varrho_{M \times N}) \\
&= (D \otimes \varphi_{M \times N}) \circ (D \otimes \Pi_D^L \otimes M \times N) \circ (\delta_D \otimes M \times N) \circ \varrho_{M \times N} \\
&= \varrho_{M \times N}.
\end{aligned}$$

To prove (yd2), using similar technics and results together with (126), (127) and the condition (yd2) referred to M and N we get:

$$\begin{aligned}
& (\mu_D \otimes M \times N) \circ (D \otimes r_{M \times N}) \circ ((\varrho_{M \times N} \circ \varphi_{M \times N}) \otimes D) \circ (D \otimes s_{M \times N}) \circ (\delta_D \otimes M \times N) \\
&= (\mu_D \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes N) \circ (D \otimes M \otimes r_N) \circ (((\mu_D \otimes \nabla_{M \otimes N}) \circ (D \otimes r_M \otimes N) \\
&\quad \circ (\varrho_M \otimes \varrho_N) \circ \nabla_{M \otimes N} \circ (\varphi_M \otimes \varphi_N) \circ (D \otimes s_M \otimes N) \circ (\delta_D \otimes i_{M \otimes N})) \otimes D) \\
&\quad \circ (D \otimes ((p_{M \otimes N} \otimes D) \circ (M \otimes s_N) \circ (s_M \otimes N))) \circ (\delta_D \otimes i_{M \otimes N}) \\
&= (\mu_D \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes N) \circ ((\varrho_M \circ \varphi_M) \otimes ((\mu_D \otimes N) \circ (D \otimes r_N) \circ ((\varrho_N \circ \varphi_N) \otimes D) \\
&\quad \circ (D \otimes s_N) \circ (\delta_D \otimes N))) \circ (D \otimes s_M \otimes N) \circ (\delta_D \otimes i_{M \otimes N}) \\
&= (\mu_D \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes N) \circ ((\varrho_M \circ \varphi_M) \otimes ((\mu_D \otimes \varphi_N) \circ (D \otimes t_{D,D} \otimes N) \circ (\delta_D \otimes \varrho_N))) \\
&\quad \circ (D \otimes s_M \otimes N) \circ (\delta_D \otimes i_{M \otimes N}) \\
&= (\mu_D \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes \varphi_N) \circ (((\mu_D \otimes M) \circ (D \otimes r_M) \circ ((\varrho_M \circ \varphi_M) \otimes D) \circ (D \otimes s_M) \\
&\quad \circ (\delta_D \otimes M)) \otimes t_{D,D} \otimes N) \circ (D \otimes s_M \otimes \varrho_N) \circ (\delta_D \otimes i_{M \otimes N}) \\
&= (\mu_D \otimes p_{M \otimes N}) \circ (D \otimes r_M \otimes \varphi_N) \circ (((\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \varrho_M)) \otimes t_{D,D} \otimes N) \\
&\quad \circ (D \otimes s_M \otimes \varrho_N) \circ (\delta_D \otimes i_{M \otimes N}) \\
&= (\mu_D \otimes p_{M \otimes N}) \circ (D \otimes ((D \otimes \varphi_M \otimes N) \circ (t_{D,D} \otimes M \otimes N) \circ (D \otimes r_M \otimes \varphi_N))) \\
&\quad \circ (\mu_D \otimes D \otimes M \otimes t_{D,D} \otimes N) \circ (D \otimes ((t_{D,D} \otimes s_M \otimes D \otimes N) \circ (D \otimes t_{D,D} \otimes M \otimes D \otimes N) \\
&\quad \circ (\delta_D \otimes \varrho_M \otimes \varrho_N))) \circ (\delta_D \otimes i_{M \otimes N}) \\
&= (\mu_D \otimes p_{M \otimes N}) \circ (D \otimes ((\mu_D \otimes \varphi_M \otimes \varphi_N) \circ (D \otimes t_{D,D} \otimes s_M \otimes N) \circ (t_{D,D} \otimes t_{D,D} \otimes M \otimes N)
\end{aligned}$$

$$\begin{aligned}
& \circ (D \otimes t_{D,D} \otimes r_M \otimes N) \circ (\delta_D \otimes \varrho_M \otimes \varrho_N))) \circ (\delta_D \otimes i_{M \otimes N}) \\
& = (\mu_D \otimes (p_{M \otimes N} \circ \varphi_{M \otimes N})) \circ (D \otimes t_{D,D} \otimes M \otimes N) \circ (\delta_D \otimes (\varrho_{M \otimes N} \circ i_{M \otimes N})) \\
& = (\mu_D \otimes \varphi_{M \times N}) \circ (D \otimes t_{D,D} \otimes M \times N) \circ (\delta_D \otimes \varrho_{M \times N}).
\end{aligned}$$

□

We proceed now to state and prove the main result of this work, giving an explicit description of all the required components of the monoidal structure for ${}^D_D\mathcal{YD}$.

Theorem 2.22. Let D be a WBHA with invertible antipode. Then ${}^D_D\mathcal{YD}$ is a non-strict monoidal category.

Proof. Given $(M, \varphi_M, \varrho_M)$ and $(N, \varphi_N, \varrho_N)$ two objects in ${}^D_D\mathcal{YD}$ we define as its product $M \times N$ the image of the idempotent $\nabla_{M \otimes N}$, that by Proposition 2.21 is a left-left Yetter-Drinfeld module with associated weak operator the one defined in Proposition 2.20.

The base object is $D_L = \text{Im}(\Pi_D^L)$; with left D -(co)module structure

$$\varphi_{D_L} = p_L \circ \mu_D \circ (D \otimes i_L), \quad \varrho_{D_L} = (D \otimes p_L) \circ \delta_D \circ i_L, \quad (132)$$

where $p_L : D \rightarrow D_L$ and $i_L : D_L \rightarrow D$ are the morphisms such that $\Pi_D^L = i_L \circ p_L$ and $p_L \circ i_L = \text{id}_{D_L}$.

It holds that $(r_{D_L}, r'_{D_L}, s_{D_L}, s'_{D_L})$ is a (D_L, D) -WO compatible with the (co)module structures of D_L , being

$$\begin{aligned}
r_{D_L} &:= (D \otimes p_L) \circ t_{D,D} \circ (i_L \otimes D); & r'_{D_L} &:= (p_L \otimes D) \circ t'_{D,D} \circ (D \otimes i_L) \\
s_{D_L} &:= (p_L \otimes D) \circ t_{D,D} \circ (D \otimes i_L); & s'_{D_L} &:= (D \otimes p_L) \circ t'_{D,D} \circ (i_L \otimes D).
\end{aligned}$$

The triple $(D_L, \varphi_{D_L}, \varrho_{D_L})$ satisfies (yd1) and (yd2) because it corresponds to the particular case of the projection $(D, \text{id}_D, \text{id}_D)$ over D [[5], Definition 2.7, Proposition 2.19] and then $(D_L, \varphi_{D_L}, \varrho_{D_L})$ is in ${}^D_D\mathcal{YD}$.

The unit constrains are:

$$\mathfrak{l}_M = \varphi_M \circ (i_L \otimes M) \circ i_{D_L \otimes M} : D_L \times M \rightarrow M, \quad (133)$$

$$\mathfrak{r}_M = \varphi_M \circ s'_M \circ (M \otimes (\overline{\Pi}_D^L \circ i_L)) \circ i_{M \otimes D_L} : M \times D_L \rightarrow M. \quad (134)$$

These morphisms are isomorphisms with inverses:

$$\mathfrak{l}_M^{-1} = p_{D_L \otimes M} \circ (p_L \otimes \varphi_M) \circ ((\delta_D \circ \eta_D) \otimes M) : M \rightarrow D_L \times M, \quad (135)$$

$$\mathfrak{r}_M^{-1} = p_{M \otimes D_L} \circ (\varphi_M \otimes p_L) \circ (D \otimes s_M) \circ ((\delta_D \circ \eta_D) \otimes M) : M \rightarrow M \times D_L, \quad (136)$$

and they are actually morphisms of ${}^D_D\mathcal{YD}$. We write the proof for one of the required equalities, the remaining being analogous. In fact:

$$\begin{aligned}
& r_M \circ (\mathfrak{r}_M \otimes D) \\
& = r_M \circ (((\varphi_M \circ s'_M \circ (M \otimes (\overline{\Pi}_D^L \circ i_L)) \circ i_{M \otimes D_L})) \otimes D) \\
& = (D \otimes \varphi_M) \circ (t_{D,D} \otimes M) \circ (D \otimes r_M) \circ (s'_M \otimes D) \circ (M \otimes (\overline{\Pi}_D^L \circ i_L) \otimes D) \circ (i_{M \otimes D_L} \otimes D) \\
& = (D \otimes (\varphi_M \circ s'_M)) \circ (r_M \otimes D) \circ (M \otimes t_{D,D}) \circ (M \otimes (\overline{\Pi}_D^L \circ i_L) \otimes D) \circ (i_{M \otimes D_L} \otimes D) \\
& = (D \otimes (\varphi_M \circ s'_M \circ (M \otimes (\overline{\Pi}_D^L \circ i_L)))) \circ (r_M \otimes D_L) \circ (M \otimes r_{D_L}) \circ (i_{M \otimes D_L} \otimes D) \\
& = (D \otimes (\varphi_M \circ s'_M \circ (M \otimes (\overline{\Pi}_D^L \circ i_L)))) \circ (D \otimes \nabla_{M \otimes D_L}) \circ (r_M \otimes D_L) \circ (M \otimes r_{D_L}) \\
& \quad \circ (i_{M \otimes D_L} \otimes D) \\
& = (D \otimes \mathfrak{r}_M) \circ r_{M \times D_L}.
\end{aligned}$$

If M, N, P are objects in the category ${}^D\mathcal{YD}$, the associativity constraints are defined by

$$\mathbf{a}_{M,N,P} = p_{(M \times N) \otimes P} \circ (p_{M \otimes N} \otimes P) \circ (M \otimes i_{N \otimes P}) \circ i_{M \otimes (N \times P)} : M \times (N \times P) \rightarrow (M \times N) \times P. \quad (137)$$

Its inverse is

$$\mathbf{a}_{M,N,P}^{-1} = p_{M \otimes (N \times P)} \circ (M \otimes p_{N \otimes P}) \circ (i_{M \otimes N} \otimes P) \circ i_{(M \times N) \otimes P} : (M \times N) \times P \rightarrow M \times (N \times P). \quad (138)$$

Using (129), (128) and Lemma 2.18 we check that they are morphisms of left-left Yetter-Drinfeld modules, and in turn this fact allows us to prove the triangle and the pentagon axioms.

As far as tensor products of morphisms in ${}^D\mathcal{YD}$ is concerned, if $\gamma : M \rightarrow M'$ and $\phi : N \rightarrow N'$ are morphisms in the category, we define

$$\gamma \times \phi = p_{M' \times N'} \circ (\gamma \otimes \phi) \circ i_{M \otimes N} : M \times N \rightarrow M' \times N', \quad (139)$$

which is a morphism in ${}^D\mathcal{YD}$ and

$$(\gamma' \times \phi') \circ (\gamma \times \phi) = (\gamma' \circ \gamma) \times (\phi' \circ \phi), \quad (140)$$

where $\gamma' : M' \rightarrow M''$ and $\phi' : N' \rightarrow N''$ are morphisms in ${}^D\mathcal{YD}$. \square

Remark 2.23. In the particular case where the category \mathcal{C} is braided with braiding c and we take $(c_{M,D}, c_{M,D}^{-1}, c_{D,M}, c_{D,M}^{-1})$ as the (M, D) -WO, the formal properties of the braiding simplify the calculations, but it is important to note that the global definition of the braiding is not an essential component in the notion of Yetter-Drinfeld module.

3. PROJECTIONS AND YETTER-DRINFELD MODULES

In this section we illustrate the preceding definitions with a family of examples, those coming from projections. These examples are especially relevant for various reasons. One of them lies on its physics motivations. In a braided category the bosonization introduced by Majid in [14] induces examples of projections. On the other hand, the Radford theory shows the key role that projections play in the theory of Yetter-Drinfeld modules.

We briefly recall the definition and main properties of projections of WBHA. The details can be found in [[5], Section 1].

Definition 3.1. Let D, B be WBHA. A projection for D is a triple (B, f, g) where $f : D \rightarrow B$ and $g : B \rightarrow D$ are morphisms of WBHA such that $g \circ f = id_D$; and satisfying the following equalities:

- (i) $(B \otimes (f \circ g)) \circ t_{B,B} = t_{B,B} \circ ((f \circ g) \otimes B).$
- (ii) $((f \circ g) \otimes B) \circ t_{B,B} = t_{B,B} \circ (B \otimes (f \circ g)).$

A morphism between two projections (B, f, g) and (B', f', g') associated to D is a morphism of WBHA $h : B \rightarrow B'$ such that $h \circ f = f'$ and $g' \circ h = g$. The set of projections associated to D and morphisms of projections is a category, which we will denote by $\mathcal{Proj}(D)$.

Remark 3.2. Notice that simultaneous verification of the conditions (i) and (ii) in Definition 3.1 for $t_{B,B}$ is equivalent to its verification for $t'_{B,B}$.

Proposition 3.3. Let D be a WBHA and let (B, f, g) be an object in $\mathcal{Proj}(D)$. The morphism $q_D^B : B \rightarrow B$, defined as

$$q_D^B := id_B \wedge (f \circ \lambda_D \circ g),$$

is an idempotent.

As a consequence there are an object B_D , an epimorphism $p_D^B : B \rightarrow B_D$, and a monomorphism $i_D^B : B_D \rightarrow B$ such that $q_D^B = i_D^B \circ p_D^B$ and $p_D^B \circ i_D^B = id_{B_D}$. Moreover $(B_D, \eta_{B_D} = p_D^B \circ \eta_B, \mu_{B_D} = p_D^B \circ \mu_B \circ (i_D^B \otimes i_D^B))$ is an algebra and $(B_D, \varepsilon_{B_D} = \varepsilon_B \circ i_D^B, \delta_{B_D} = (p_D^B \otimes p_D^B) \circ \delta_B \circ i_D^B)$ is a coalgebra in \mathcal{C} and $(B_D, \varphi_{B_D} = p_D^B \circ \mu_B \circ (f \otimes i_D^B))$ is a left D -module and $(B_D, \rho_{B_D} = (g \otimes p_D^B) \circ \delta_B \circ i_D^B)$ is a left D -comodule.

Proof. See [[5], Propositions 2.11, 2.13 and 2.17]. \square

Proposition 3.4. Let D be a WBHA and $(B, f, g) \in |\mathcal{P}roj(D)|$. We define:

$$\begin{aligned} r_{B_D} &:= (g \otimes p_D^B) \circ t_{B,B} \circ (i_D^B \otimes f); & r'_{B_D} &:= (p_D^B \otimes g) \circ t'_{B,B} \circ (f \otimes i_D^B), \\ s_{B_D} &:= (p_D^B \otimes g) \circ t_{B,B} \circ (f \otimes i_D^B); & s'_{B_D} &:= (g \otimes p_D^B) \circ t'_{B,B} \circ (i_D^B \otimes f). \end{aligned}$$

It holds that the quadruple $(r_{B_D}, r'_{B_D}, s_{B_D}, s'_{B_D})$ is a (B_D, D) -WO compatible with the (co)module structure defined for B_D in Proposition 3.3.

Proof. On each condition, just some parts are proved to illustrate the technics applied, the remaining being analogous.

On the condition (c1) we check (c1-1) explicitly:

$$\begin{aligned} & (D \otimes r_{B_D}) \circ (r_{B_D} \otimes D) \circ (B_D \otimes t_{D,D}) \\ &= (D \otimes ((g \otimes p_D^B) \circ t_{B,B})) \circ (((g \otimes q_D^B) \circ t_{B,B}) \otimes B) \circ (i_D^B \otimes ((f \otimes f) \circ t_{D,D})) \\ &= (g \otimes g \otimes p_D^B) \circ (B \otimes t_{B,B}) \circ (t_{B,B} \otimes B) \circ (i_D^B \otimes (t_{B,B} \circ (f \otimes f))) \\ &= (((g \otimes g) \circ t_{B,B}) \otimes p_D^B) \circ (B \otimes t_{B,B}) \circ (t_{B,B} \otimes B) \circ (i_D^B \otimes f \otimes f) \\ &= (t_{D,D} \otimes B_D) \circ (g \otimes g \otimes p_D^B) \circ (B \otimes t_{B,B}) \circ (B \otimes q_D^B \otimes B) \circ (t_{B,B} \otimes B) \circ (i_D^B \otimes f \otimes f) \\ &= (t_{D,D} \otimes B_D) \circ (D \otimes r_{B_D}) \circ (r_{B_D} \otimes D). \end{aligned}$$

In the preceding calculations, the first and the last equalities hold by the definition of r_{B_D} , on the second one we use [[5], Lemma 2.16] and the fact that f is a morphism of WBHA. The third one follows because of $t_{B,B}$ verifies the Yang-Baxter equation, and the fourth one uses [[5], Lemma 2.16] together with the character of morphism of WBHA for g .

The proof for the condition (c2) is similar, but using the equality (3) instead of the verification of the Yang-Baxter equation.

For (c3-1), using that B is a WBHA, the definition of projection, [[5], Lemma 2.16], (b2-3) and the equality (20) we have:

$$\begin{aligned} & \nabla_{r_{B_D}} \\ &= r_{B_D} \circ r'_{B_D} \\ &= (p_D^B \otimes g) \circ t'_{B,B} \circ ((f \circ g) \otimes q_D^B) \circ t_{B,B} \circ (i_D^B \otimes f) \\ &= (p_D^B \otimes g) \circ \nabla_{B,B} \circ (i_D^B \otimes f) \\ &= (p_D^B \otimes \varepsilon_B \otimes g) \circ (\nabla_{B,B} \otimes B) \circ (i_D^B \otimes (\delta_B \circ f)) \\ &= (\varepsilon_B \otimes p_D^B \otimes g) \circ (t_{B,B} \otimes B) \circ (i_D^B \otimes (\delta_B \circ f)) \\ &= (\varepsilon_B \otimes B_D \otimes D) \circ (r_{B_D} \otimes D) \circ (B_D \otimes \delta_D). \end{aligned}$$

Arguing analogously we obtain that

$$\nabla_{r_{B_D}} = (\mu_D \otimes B_D) \circ (D \otimes (r_{B_D} \circ (B_D \otimes \eta_D))).$$

For the condition (c4-1), by the definition of projection, [[5], Lemma 2.16] and the properties of the weak operator $t_{B,B}$ we know that:

$$\begin{aligned} & (\mu_D \otimes B_D) \circ (D \otimes r_{B_D}) \circ (r_{B_D} \otimes D) \\ &= (\mu_D \otimes B_D) \circ (g \otimes g \otimes p_D^B) \circ (B \otimes t_{B,B}) \circ (B \otimes q_D^B \otimes B) \circ (t_{B,B} \otimes B) \circ (i_D^B \otimes f \otimes f) \\ &= (g \otimes p_D^B) \circ (\mu_B \otimes B) \circ (B \otimes t_{B,B}) \circ (t_{B,B} \otimes B) \circ (i_D^B \otimes f \otimes f) \end{aligned}$$

$$\begin{aligned}
&= (g \otimes p_D^B) \circ t_{B,B} \circ (i_D^B \otimes (\mu_B \circ (f \otimes f))) \\
&= r_{B_D} \circ (B_D \otimes \mu_D).
\end{aligned}$$

Finally, the condition (c5) follows because f and g are morphisms of WBHA and by [[2], Proposition 2.12].

In order to see the compatibility with the (co)module structures of B_D , we just state explicitly one of the required equalities to illustrate the technics. We can write:

$$\begin{aligned}
&(D \otimes \varphi_{B_D}) \circ (t_{D,D} \otimes B_D) \circ (D \otimes r_{B_D}) \\
&= (D \otimes (p_D^B \circ \mu_B \circ (f \otimes q_D^B))) \circ (t_{D,D} \otimes B) \circ (D \otimes g \otimes B) \circ (D \otimes (t_{B,B} \circ (i_D^B \otimes f))) \\
&= (g \otimes (p_D^B \circ \mu_B)) \circ (t_{B,B} \otimes B) \circ (B \otimes t_{B,B}) \circ (f \otimes i_D^B \otimes f) \\
&= (g \otimes p_D^B) \circ t_{B,B} \circ (\mu_B \otimes B) \circ (f \otimes i_D^B \otimes f) \\
&= (g \otimes p_D^B) \circ t_{B,B} \circ ((i_D^B \circ p_D^B \circ \mu_B) \otimes B) \circ (f \otimes i_D^B \otimes f) \\
&= r_{B_D} \circ (\varphi_{B_D} \otimes D).
\end{aligned}$$

The remaining equalities are analogous. \square

The above disquisitions allow to state one of the main results of this section:

Proposition 3.5. Let D be a WBHA and $(B, f, g) \in |\mathcal{P}roj(D)|$. With the notation of Proposition 3.3, $(B_D, \varphi_{B_D}, \varrho_{B_D})$ is in ${}^D_D\mathcal{YD}$.

Proof. We have already shown that the quadruple $(r_{B_D}, r'_{B_D}, s_{B_D}, s'_{B_D})$ defined in Proposition 3.4 is a (B_D, D) -WO compatible with the (co)-module structure. For the conditions (yd1) and (yd2) see [[5], Proposition 1.19]. \square

Remark 3.6. This example arising from projections of WBHA also suggests that the requirement introduced in part (ii) of Definition 2.5 is natural in the sense that it is automatically satisfied in the case of this generic example. Actually, by definition of morphism between projections over D , given such a morphism $\alpha : B \rightarrow B'$ between (B, f, g) and (B', f', g') (see Definition 3.1) it induces a (co)module morphism $\alpha_D : B_D \rightarrow B'_D$ such that $i_D^{B'} \circ \alpha_D = \alpha \circ i_D^B$. Then we have

$$\begin{aligned}
&r_{B'_D} \circ (\alpha_D \otimes D) \\
&= (g' \otimes p_D^{B'}) \circ t_{B',B'} \circ ((i_D^{B'} \circ \alpha_D) \otimes f') \\
&= (g' \otimes p_D^{B'}) \circ t_{B',B'} \circ ((\alpha \circ i_D^B) \otimes (\alpha \circ f)) \\
&= ((g' \circ \alpha) \otimes (p_D^{B'} \circ \alpha)) \circ t_{B,B} \circ (i_D^B \otimes f) \\
&= (g \otimes (\alpha_D \circ p_D^B)) \circ t_{B,B} \circ (i_D^B \otimes f) \\
&= (D \otimes \alpha_D) \circ r_{B_D}.
\end{aligned}$$

We would obtain similarly the analogous results for r'_{B_D} , s_{B_D} and s'_{B_D} instead of r_{B_D} , so any morphism in $\mathcal{P}roj(D)$ induces naturally a morphism in ${}^D_D\mathcal{YD}$.

On these examples coming from projections the construction of the weak operator is based on the weak Yang-Baxter operator $t_{B,B}$ and its properties. We will finish this section seeing a link between the notions of weak Yang-Baxter operator and weak entwining structure, being the last one relevant, for example, in order to give a characterization of weak cleft extensions in terms of weak Galois extensions with normal basis, as can be found in [4]. To do so, the definition of invertible weak entwining is briefly recalled (see [4] for details).

Definition 3.7. A right-right weak entwining structure is a triple (A, C, Ψ_{RR}) where A is an algebra, C a coalgebra and $\Psi_{RR} : C \otimes A \rightarrow A \otimes C$ is a morphism that satisfies:

$$(\mu_A \otimes C) \circ (A \otimes \Psi_{RR}) \circ (\Psi_{RR} \otimes A) = \Psi_{RR} \circ (C \otimes \mu_A), \quad (141)$$

$$\Psi_{RR} \circ (C \otimes \eta_A) = (e_{RR} \otimes C) \circ \delta_C, \quad (142)$$

$$(A \otimes \delta_C) \circ \Psi_{RR} = (\Psi_{RR} \otimes C) \circ (C \otimes \Psi_{RR}) \circ (\delta_C \otimes A), \quad (143)$$

$$(A \otimes \varepsilon_C) \circ \Psi_{RR} = \mu_A \circ (e_{RR} \otimes A). \quad (144)$$

with $e_{RR} = (A \otimes \varepsilon_C) \circ \Psi_{RR} \circ (C \otimes \eta_A)$. Similarly we can define a left-left weak entwining structure (A, C, Ψ_{LL}) for an algebra A , a coalgebra C and a morphism $\Psi_{LL} : A \otimes C \rightarrow C \otimes A$ that verifies similar equalities to the previous ones with $e_{LL} = (\varepsilon_C \otimes A) \circ \Psi_{LL} \circ (\eta_A \otimes C)$.

3.8. Let (A, C, Ψ_{RR}) be a right-right weak entwining structure. Define $\Delta_{RR} : A \otimes C \rightarrow A \otimes C$ by $\Delta_{RR} = (\mu_A \otimes C) \circ (A \otimes \Psi_{RR}) \circ (A \otimes C \otimes \eta_A)$. This morphism is idempotent and so is the morphism $\nabla_{RR} : C \otimes A \rightarrow C \otimes A$ defined by $\nabla_{RR} = (C \otimes A \otimes \varepsilon_C) \circ (C \otimes \Psi_{RR}) \circ (\delta_C \otimes A)$. The corresponding idempotent morphisms for a left-left weak entwining structure will be denoted by Δ_{LL} and ∇_{LL} .

Definition 3.9. Let A be an algebra, C a coalgebra and $\Psi_{RR} : C \otimes A \rightarrow A \otimes C$ and $\Psi_{LL} : A \otimes C \rightarrow C \otimes A$ morphisms in \mathcal{C} . We say that $(C, A, \Psi_{RR}, \Psi_{LL})$ is an invertible weak entwining structure if the following conditions hold:

- (i) (A, C, Ψ_{RR}) is a right-right weak entwining structure and (A, C, Ψ_{LL}) is a left-left weak entwining structure.
- (ii) $\Psi_{LL} \circ \Psi_{RR} = \Delta_{LL}$ and $\Psi_{RR} \circ \Psi_{LL} = \Delta_{RR}$

The relation between weak Yang-Baxter operators and invertible weak entwining structures can be expressed in terms of weak operators as follows:

Proposition 3.10. With the notation of Proposition 3.4, it holds that

- (i) $(B_D, D, s_{B_D}, s'_{B_D})$ is an invertible weak entwining structure.
- (ii) $(B_D, D, r'_{B_D}, r_{B_D})$ is an invertible weak entwining structure.

Proof. We prove part (i). Let's see that (B_D, D, s_{B_D}) is a right-right weak entwining structure. First of all, it was already demonstrated that the quadruple $(r_{B_D}, r'_{B_D}, s_{B_D}, s'_{B_D})$ is a (B_D, D) -WO, so we know that (143) holds. On the other hand, using that $(B, f, g) \in |\mathcal{P}roj(D)|$ and the properties of the weak operator $t_{B,B}$ and [[5], Lemma 2.16] we obtain (141). Indeed:

$$\begin{aligned} & (\mu_{B_D} \otimes D) \circ (B_D \otimes s_{B_D}) \circ (s_{B_D} \otimes B_D) \\ &= ((p_D^B \circ \mu_B \circ ((i_D^B \circ p_D^B) \otimes (i_D^B \circ p_D^B))) \otimes g) \circ (B \otimes t_{B,B}) \circ (B \otimes (f \circ g) \otimes B) \circ (t_{B,B} \otimes B) \circ (f \otimes i_D^B \otimes i_D^B) \\ &= ((p_D^B \circ \mu_B) \otimes g) \circ (B \otimes t_{B,B}) \circ (t_{B,B} \otimes B) \circ (f \otimes i_D^B \otimes i_D^B) \\ &= (p_D^B \otimes g) \circ t_{B,B} \circ (B \otimes \mu_B) \circ (f \otimes i_D^B \otimes i_D^B) \\ &= s_{B_D} \circ (D \otimes \mu_{B_D}). \end{aligned}$$

To show (142), note first that in this case $e_{RR} = (p_D^B \otimes \varepsilon_B) \circ t_{B,B} \circ (f \otimes \eta_B)$, so

$$\begin{aligned} & (e_{RR} \otimes D) \circ \delta_D \\ &= (p_D^B \otimes \varepsilon_B \otimes D) \circ (t_{B,B} \otimes D) \circ (f \otimes \eta_B \otimes D) \circ \delta_D \\ &= (p_D^B \otimes \varepsilon_B \otimes g) \circ (\nabla_{B,B} \otimes B) \circ (\eta_B \otimes f \otimes f) \circ \delta_D \\ &= (p_D^B \otimes \varepsilon_B \otimes g) \circ (\nabla_{B,B} \otimes B) \circ (\eta_B \otimes (\delta_B \circ f)) \\ &= (p_D^B \otimes \varepsilon_B \otimes g) \circ (B \otimes \delta_B) \circ \nabla_{B,B} \circ (\eta_B \otimes f) \\ &= (p_D^B \otimes g) \circ t_{B,B} \circ (f \otimes i_D^B) \circ (D \otimes \eta_{B_D}) \\ &= s_{B_D} \circ (D \otimes \eta_{B_D}). \end{aligned}$$

In the above calculations, the first and the fifth equalities are just the definition of e_{RR} , the second one uses (17) and $g \circ f = id_D$; the third one follows because of f is a coalgebra morphism, and the fourth relies on (b2-3). By similar arguments we get (144).

The proof showing that (B_D, D, s'_{B_D}) is a left-left weak entwining is analogous.

Finally we use similar properties to see that $s_{B_D} \circ s'_{B_D} = \Delta_{RR}$:

$$\begin{aligned}
& s_{B_D} \circ s'_{B_D} \\
&= (p_D^B \otimes g) \circ t_{B,B} \circ ((f \circ g) \otimes (i_D^B \circ p_D^B)) \circ t'_{B,B} \circ (i_D^B \otimes f) \\
&= (p_D^B \otimes g) \circ \nabla_{B,B} \circ (i_D^B \otimes f) \\
&= (p_D^B \otimes g) \circ \nabla_{B,B} \circ (\mu_B \otimes B) \circ (i_D^B \otimes \eta_B \otimes f) \\
&= (p_D^B \otimes g) \circ (\mu_B \otimes B) \circ (B \otimes \nabla_{B,B}) \circ (i_D^B \otimes \eta_B \otimes f) \\
&= (p_D^B \otimes D) \circ (\mu_B \otimes B) \circ (i_D^B \otimes (i_D^B \circ p_D^B) \otimes g) \circ (B \otimes t_{B,B}) \circ (B_D \otimes f \otimes (i_D^B \circ \eta_{B_D})) \\
&= (\mu_{B_D} \otimes D) \circ (B_D \otimes s_{B_D}) \circ (B_D \otimes D \otimes \eta_{B_D}) \\
&= \Delta_{RR}.
\end{aligned}$$

It can be checked similarly that $s'_{B_D} \circ s_{B_D} = \Delta_{LL}$. \square

4. ADJOINT (CO)ACTIONS AND YETTER-DRINFELD MODULES

In the theory of Hopf algebras it is a well-known fact that, if H is a Hopf algebra in an strict braided monoidal category with braid c , the triple (H, φ_H, δ_H) is an object of ${}^H_H\mathcal{YD}$ where $\varphi_H : H \otimes H \rightarrow H$ denotes the adjoint action defined by

$$\varphi_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

Also, the triple (H, μ_H, ϱ_H) is an object of ${}^H_H\mathcal{YD}$ where $\varrho_H : H \rightarrow H \otimes H$ denotes the adjoint coaction defined by

$$\varrho_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes \lambda_H) \circ \delta_H.$$

Unfortunately, in the weak setting, the previous assertions are not true in general (see [6]). Indeed, being H a weak Hopf algebra in \mathcal{C} , the pair (H, φ_H) is not in general a left H -module because the unit condition can fail, i.e.

$$\varphi_H \circ (\eta_H \otimes H) = \mu_H \circ (H \otimes (\lambda_H \circ \Pi_H^L)) \circ \delta_H \neq id_H,$$

and for the adjoint coaction the counit condition may be untrue because

$$(\varepsilon_H \otimes H) \circ \varrho_H = \mu_H \circ (H \otimes (\Pi_H^L \circ \lambda_H)) \circ \delta_H \neq id_H.$$

In this section we shall show that for every WBHA D the adjoint action and the adjoint coaction induce idempotent morphisms and as a consequence, using the factorizations of these idempotents, it is possible to construct new examples of objects in the category ${}^D_D\mathcal{YD}$ defined in the second section of this paper. Obviously, if H is a Hopf algebra, the idempotents associated to the adjoint action and coaction are identities and we recover the classical results.

Proposition 4.1. Let D be a WBHA in \mathcal{C} . Let $\varphi_D : D \otimes D \rightarrow D$ and $\varrho_D : D \rightarrow D \otimes D$ be the morphisms defined by

$$\varphi_D = \mu_D \circ (\mu_D \otimes \lambda_D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D)$$

and

$$\varrho_D = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes \lambda_D) \circ \delta_D.$$

Then

$$\omega_D^a = \varphi_D \circ (\eta_D \otimes D) : D \rightarrow D,$$

$$\omega_D^c = (\varepsilon_D \otimes D) \circ \varrho_D : D \rightarrow D$$

are idempotent morphisms in \mathcal{C} and

$$\omega_D^a = \mu_D \circ (D \otimes (\lambda_D \circ \Pi_D^L)) \circ \delta_D,$$

$$\omega_D^c = \mu_D \circ (D \otimes (\Pi_D^L \circ \lambda_D)) \circ \delta_D.$$

Proof. We prove the idempotent condition for ω_D^a . The proof for ω_D^c is analogous and we leave the details to the reader.

$$\begin{aligned} & \omega_D^a \circ \omega_D^a \\ &= \mu_D \circ (\mu_D \otimes (\mu_D \circ (\lambda_D \otimes \lambda_D) \circ t_{D,D})) \circ (\mu_D \otimes t_{D,D} \otimes D) \circ (D \otimes t_{D,D} \otimes t_{D,D}) \\ & \quad \circ (\delta_D \otimes \delta_D \otimes D) \circ (\eta_D \otimes \eta_D \otimes D) \\ &= \mu_D \circ (\mu_D \otimes \lambda_D) \circ (D \otimes t_{D,D}) \circ (((\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)) \otimes D) \\ & \quad \circ (\eta_D \otimes \eta_D \otimes D) \\ &= \mu_D \circ (\mu_D \otimes \lambda_D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \mu_D) \otimes D) \circ (\eta_D \otimes \eta_D \otimes D) \\ &= \omega_D^a. \end{aligned}$$

The first equality follows by (b3-2) and associativity of μ_D . The second one is a consequence of [[2], Proposition 2.20] and (b3-1). Finally, the third one follows by (b4) and the fourth one by the unit condition for μ_D .

The equalities

$$\omega_D^a = \mu_D \circ (D \otimes (\lambda_D \circ \Pi_D^L)) \circ \delta_D$$

and

$$\omega_D^c = \mu_D \circ (D \otimes (\Pi_D^L \circ \lambda_D)) \circ \delta_D$$

follow from [[2], Proposition 2.12]. \square

Examples 4.2. i) Let $D = RG$ be the groupoid algebra considered in (1) of 1.4. Then the morphisms defined in the previous Proposition are:

$$\omega_{RG}^a(\sigma) = \sigma \circ id_{t(\sigma)} = \begin{cases} \sigma & \text{if } t(\sigma) = s(\sigma) \\ 0 & \text{if } t(\sigma) \neq s(\sigma) \end{cases}$$

$$\omega_{RG}^c(\sigma) = \sigma \circ id_{s(\sigma)} = \sigma.$$

In the particular case of the groupoid algebra on n -objects with one invertible arrow between each ordered pair of objects, we obtain that RG is isomorphic to the $n \times n$ matrix $RG = M_n(R)$. The weak Hopf algebra H has the following structure. If E_{ij} denote the (i, j) - matrix unit, RG has counit given by $\varepsilon_{RG}(E_{ij}) = 1$, comultiplication by $\delta_{RG}(E_{ij}) = E_{ij} \otimes E_{ij}$ and antipode given by $\lambda_{RG}(E_{ij}) = E_{ji}$ for each $i, j = 1, \dots, n$. In this case, $\Pi_{RG}^L(E_{ij}) = E_{ii}$, $\Pi_{RG}^R(E_{ij}) = E_{jj}$ and then $RG_L = RG_R$ is the submodule of the diagonal matrices. Therefore, the image of ω_{RG}^a is RG_L .

ii) In a general setting, if D is a commutative ($\mu_D = \mu_D \circ t_{D,D}$) WBHA, then $\Pi_D^L = \overline{\Pi}_D^R$, so by (30) and (26), we have

$$\begin{aligned} & \omega_D^a \\ &= \mu_D \circ (D \otimes \Pi_D^R) \circ \delta_D \\ &= id_D \wedge \Pi_D^R \\ &= id_D, \\ & \text{and } \omega_D^c = id_D \wedge \Pi_D^L. \end{aligned}$$

In a similar way, if D is a cocommutative ($\delta_D = t_{D,D} \circ \delta_D$) WBHA, then $\Pi_D^L = \overline{\Pi}_D^L$, and $\omega_D^a = id_D \wedge \Pi_D^L$ and $\omega_D^c = id_D$.

iii) In this example we assume that \mathcal{C} is braided with braiding c . Let $A = (A, \eta_A, \mu_A, \varepsilon_A, \delta_A)$ be a separable Frobenius algebra in \mathcal{C} . Using the separability condition $\mu_A \circ \delta_A = id_A$, we get that $A \otimes A$ is a weak Hopf algebra in \mathcal{C} (see [18]) where

$$\eta_{A \otimes A} = \eta_A \otimes \eta_A, \quad \mu_{A \otimes A} = ((\mu_A \circ c_{A,A}) \otimes \mu_A) \circ (A \otimes c_{A,A} \otimes A),$$

$$\begin{aligned}\varepsilon_{A \otimes A} &= \varepsilon_A \circ \mu_A, \quad \delta_{A \otimes A} = A \otimes (\delta_A \circ \eta_A) \otimes A, \\ \lambda_{A \otimes A} &= (\varepsilon_{A \otimes A} \otimes A \otimes A) \circ (A \otimes c_{A,A} \otimes A) \circ ((\delta_A \circ \eta_A) \otimes c_{A,A}).\end{aligned}$$

For $A \otimes A$ we have

$$\Pi_{A \otimes A}^L = (((\varepsilon_A \circ \mu_A) \otimes A) \circ (A \otimes c_{A,A}) \circ (\delta_A \otimes A)) \otimes \eta_A,$$

and then

$$\begin{aligned}\omega_{A \otimes A}^a &= (A \otimes (\mu_A \circ (\mu_A \otimes A) \circ (A \otimes c_{A,A}) \circ ((\delta_A \circ \eta_A) \otimes A))), \\ \omega_{A \otimes A}^c &= ((\varepsilon_A \circ \mu_A) \otimes (\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes c_{A,A}^{-1} \otimes c_{A,A}) \circ (c_{A,A}^{-1} \otimes c_{A,A} \otimes A) \\ &\quad \circ (A \otimes (\delta_A \circ \eta_A) \otimes (c_{A,A}^{-1} \circ \delta_A)).\end{aligned}$$

- iv) Now we come back to section 3. Let D be a WBHA with invertible antipode and let (B, f, g) be an object in $\mathcal{P}roj(D)$. Then the object B_D defined in Proposition 3.3 is a WBHA [[5], Theorem 3.4] with associated weak Yang-Baxter operator

$$t_{B_D, B_D} = (\varphi_{B_D} \otimes B_D) \circ (D \otimes r_{B_D, B_D}) \circ (\rho_{B_D} \otimes B_D) : B_D \otimes B_D \rightarrow B_D \otimes B_D,$$

where $r_{B_D, B_D} = (p_D^B \otimes p_D^B) \circ t_{B, B} \circ (i_D^B \otimes i_D^B)$, and antipode

$$\lambda_{B_D} = p_D^B \circ \mu_B \circ ((f \circ g) \otimes \lambda_B) \circ \delta_B \circ i_D^B.$$

Using the equalities proved in [[5], Lemma 2.16], $\varepsilon_B \circ q_D^B = \varepsilon_B$ and $\eta_B = q_D^B \circ \eta_B$ we obtain

$$\Pi_{B_D}^L = ((\varepsilon_B \circ \mu_B) \otimes p_D^B) \circ (B \otimes t_{B, B}) \circ ((\delta_B \circ \eta_B) \otimes i_D^B) = p_D^B \circ \Pi_B^L \circ i_D^B.$$

Then,

$$\omega_{B_D}^a = p_D^B \circ (q_D^B \wedge (i_D^B \circ \lambda_{B_D} \circ p_D^B \circ \Pi_B^L)) \circ i_D^B.$$

Similarly,

$$\omega_{B_D}^c = p_D^B \circ (q_D^B \wedge (\Pi_B^L \circ i_D^B \circ \lambda_{B_D} \circ p_D^B)) \circ i_D^B.$$

Proposition 4.3. Let D be a WBHA in \mathcal{C} . Let $\varphi_D : D \otimes D \rightarrow D$ and $\varrho_D : D \rightarrow D \otimes D$ be the morphisms defined in Proposition 4.1. Then the following assertions hold:

$$t_{D, D} \circ (\varphi_D \otimes D) = (D \otimes \varphi_D) \circ (t_{D, D} \otimes D) \circ (D \otimes t_{D, D}), \quad (145)$$

$$t_{D, D} \circ (D \otimes \varphi_D) = (\varphi_D \otimes D) \circ (D \otimes t_{D, D}) \circ (t_{D, D} \otimes D), \quad (146)$$

$$(\varrho_D \otimes D) \circ t_{D, D} = (D \otimes t_{D, D}) \circ (t_{D, D} \otimes D) \circ (D \otimes \varrho_D), \quad (147)$$

$$(D \otimes \varrho_D) \circ t_{D, D} = (t_{D, D} \otimes D) \circ (D \otimes t_{D, D}) \circ (\varrho_D \otimes D), \quad (148)$$

$$t'_{D, D} \circ (\varphi_D \otimes D) = (D \otimes \varphi_D) \circ (t'_{D, D} \otimes D) \circ (D \otimes t'_{D, D}), \quad (149)$$

$$t'_{D, D} \circ (D \otimes \varphi_D) = (\varphi_D \otimes D) \circ (D \otimes t'_{D, D}) \circ (t'_{D, D} \otimes D), \quad (150)$$

$$(\varrho_D \otimes D) \circ t'_{D, D} = (D \otimes t'_{D, D}) \circ (t'_{D, D} \otimes D) \circ (D \otimes \varrho_D), \quad (151)$$

$$(D \otimes \varrho_D) \circ t'_{D,D} = (t'_{D,D} \otimes D) \circ (D \otimes t'_{D,D}) \circ (\varrho_D \otimes D). \quad (152)$$

Proof. We write by way of example the proof for (149); the others being analogous.

$$\begin{aligned} & t'_{D,D} \circ (\varphi_D \otimes D) \\ &= (D \otimes \mu_D) \circ (t'_{D,D} \otimes \mu_D) \circ (D \otimes t'_{D,D} \otimes \lambda_D) \circ (D \otimes D \otimes t'_{D,D}) \circ (D \otimes t_{D,D} \otimes D) \\ & \quad \circ (\delta_D \otimes D \otimes D) \\ &= (D \otimes \varphi_D) \circ (t'_{D,D} \otimes D) \circ (D \otimes t'_{D,D}). \end{aligned}$$

The first equality follows by (21) and by [[2], Proposition 2.12]. The second one is a consequence of (6). \square

Proposition 4.4. Let D be a WBHA in \mathcal{C} . Let $\varphi_D : D \otimes D \rightarrow D$ and $\varrho_D : D \rightarrow D \otimes D$ be the morphisms defined in Proposition 4.1. Then the following assertions hold:

$$\varphi_D \circ (D \otimes \varphi_D) = \varphi_D \circ (\mu_D \otimes D), \quad (153)$$

$$\delta_D \circ \varphi_D = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (((\mu_D \otimes \varphi_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)) \otimes \lambda_D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D), \quad (154)$$

$$(D \otimes \varrho_D) \circ \varrho_D = (\delta_D \otimes D) \circ \varrho_D, \quad (155)$$

$$\varrho_D \circ \mu_D = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (((\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \varrho_D)) \otimes \lambda_D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D). \quad (156)$$

Proof. The proof for (153) is similar to the one used to prove the idempotent character of ω_D^a removing in the equalities the morphism $\eta_D \otimes \eta_D \otimes D$.

To see (154), using that D is a WBHA and (32), we have

$$\begin{aligned} & (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (((\mu_D \otimes \varphi_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)) \otimes \lambda_D) \circ (D \otimes t_{D,D}) \\ & \quad \circ (\delta_D \otimes D) \\ &= (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (((\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)) \\ & \quad \otimes (t_{D,D} \circ (\lambda_D \otimes \lambda_D) \circ \delta_D)) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D) \\ &= (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ ((\delta_D \circ \mu_D) \otimes (\delta_D \circ \lambda_D)) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D) \\ &= \delta_D \circ \varphi_D. \end{aligned}$$

The proofs for (155) and (156) are analogous and we leave the details to the reader. \square

Proposition 4.5. Let D be a WBHA in \mathcal{C} . Let ω_D^a, ω_D^c be the idempotent morphisms defined in Proposition 4.1. Then the following assertions hold:

$$\varphi_D \circ (D \otimes \omega_D^a) = \varphi_D, \quad (157)$$

$$(D \otimes \omega_D^a) \circ \delta_D \circ \omega_D^a = \delta_D \circ \omega_D^a, \quad (158)$$

$$\varrho_D \circ (D \otimes \omega_D^c) = \varrho_D, \quad (159)$$

$$\omega_D^c \circ \mu_D \circ (D \otimes \omega_D^c) = \omega_D^c \circ \mu_D. \quad (160)$$

Proof. As in the previous results we prove (157) and (158) leaving the other equalities to the reader. The proof of (157) is a direct consequence of (153). To check (158), first note that by (157) the equality

$$(D \otimes \omega_D^a) \circ t_{D,D} \circ (\varphi_D \otimes D) = t_{D,D} \circ (\varphi_D \otimes D) \quad (161)$$

holds. Then, composing in (154) with $\eta_D \otimes D$ and $D \otimes \omega_D^a$ we have

$$\begin{aligned} & (D \otimes \omega_D^a) \circ \delta_D \circ \omega_D^a \\ &= (D \otimes \omega_D^a) \circ (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (((\mu_D \otimes \varphi_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)) \otimes \lambda_D) \\ & \quad \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D) \\ &= (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (((\mu_D \otimes \varphi_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)) \otimes \lambda_D) \circ (D \otimes t_{D,D}) \\ & \quad \circ ((\delta_D \circ \eta_D) \otimes D) \\ &= \delta_D \circ \omega_D^a. \end{aligned}$$

□

Notation 4.6. Let D be a WBHA in \mathcal{C} . Let ω_D^a, ω_D^c be the idempotent morphisms defined in Proposition 4.1. For $x \in \{a, c\}$, with $\Omega^x(D)$, $p_D^x : D \rightarrow \Omega^x(D)$, $i_D^x : \Omega^x(D) \rightarrow D$ we denote the object and the morphisms such that $\omega_D^x = i_D^x \circ p_D^x$ and $id_{\Omega^x(D)} = p_D^x \circ i_D^x$.

Proposition 4.7. Let D be a WBHA in \mathcal{C} . The following assertions hold:

- (i) The object $\Omega^a(D)$ is a left D -module with action

$$\varphi_{\Omega^a(D)} = p_D^a \circ \varphi_D \circ (D \otimes i_D^a) : D \otimes \Omega^a(D) \rightarrow \Omega^a(D)$$

and a left D -comodule with coaction

$$\rho_{\Omega^a(D)} = (D \otimes p_D^a) \circ \delta_D \circ i_D^a : \Omega^a(D) \rightarrow D \otimes \Omega^a(D).$$

- (ii) The object $\Omega^c(D)$ is a left D -module with action

$$\psi_{\Omega^c(D)} = p_D^c \circ \mu_D \circ (D \otimes i_D^c) : D \otimes \Omega^c(D) \rightarrow \Omega^c(D)$$

and a left D -comodule with coaction

$$\varrho_{\Omega^c(D)} = (D \otimes p_D^c) \circ \varrho_D \circ i_D^c : \Omega^c(D) \rightarrow D \otimes \Omega^c(D).$$

Proof. We shall prove (i). The proof for the second assertion is analogous.

Firstly note that

$$\varphi_{\Omega^a(D)} \circ (\eta_D \otimes \Omega^a(D)) = p_D^a \circ \omega_D^a \circ i_D^a = id_{\Omega^a(D)}.$$

Secondly, by (153) and (157), we have

$$\begin{aligned} & \varphi_{\Omega^a(D)} \circ (D \otimes \varphi_{\Omega^a(D)}) \\ &= p_D^a \circ \varphi_D \circ (D \otimes \omega_D^a) \circ (D \otimes \varphi_D) \circ (D \otimes D \otimes i_D^a) \\ &= p_D^a \circ \varphi_D \circ (D \otimes \varphi_D) \circ (D \otimes D \otimes i_D^a) \\ &= p_D^a \circ \varphi_D \circ (\mu_D \otimes D) \circ (D \otimes D \otimes i_D^a) \\ &= \varphi_{\Omega^a(D)} \circ (\mu_D \otimes \Omega^a(D)). \end{aligned}$$

On the other hand, trivially $(\varepsilon_D \otimes \Omega^a(D)) \circ \rho_{\Omega^a(D)} = id_{\Omega^a(D)}$. Finally, by (158) we have

$$\begin{aligned} & (D \otimes \rho_{\Omega^a(D)}) \circ \rho_{\Omega^a(D)} \\ &= (D \otimes ((D \otimes p_D^a) \circ \delta_D)) \circ (D \otimes \omega_D^a) \circ \delta_D \circ \omega_D^a \circ i_D^a \\ &= (D \otimes ((D \otimes p_D^a) \circ \delta_D)) \circ \delta_D \circ i_D^a \\ &= (\delta_D \otimes \Omega^a(D)) \circ \rho_{\Omega^a(D)}. \end{aligned}$$

□

Proposition 4.8. Let D be a WBHA in \mathcal{C} . The following assertions hold:

- (i) The object $(\Omega^a(D), \varphi_{\Omega^a(D)}, \rho_{\Omega^a(D)})$ is in ${}^D_B\mathcal{YD}$.
- (ii) The object $(\Omega^c(D), \psi_{\Omega^c(D)}, \varrho_{\Omega^c(D)})$ is in ${}^D_B\mathcal{YD}$.

Proof. First note that by (b3) of Definition 1.5, the properties of the weak operator $t_{D,D}$ and [[2], Propositions 2.11, 2.12], the following identities hold

$$t_{D,D} \circ (\omega_D^x \otimes D) = (D \otimes \omega_D^x) \circ t_{D,D}, \quad t_{D,D} \circ (D \otimes \omega_D^x) = (\omega_D^x \otimes D) \circ t_{D,D}, \quad (162)$$

$$t'_{D,D} \circ (\omega_D^x \otimes D) = (D \otimes \omega_D^x) \circ t'_{D,D}, \quad t'_{D,D} \circ (D \otimes \omega_D^x) = (\omega_D^x \otimes D) \circ t'_{D,D}, \quad (163)$$

$$\nabla_{D,D} \circ (\omega_D^x \otimes D) = (\omega_D^x \otimes D) \circ \nabla_{D,D}, \quad \nabla_{D,D} \circ (D \otimes \omega_D^x) = (D \otimes \omega_D^x) \circ \nabla_{D,D}, \quad (164)$$

for $x \in \{a, c\}$.

Following the notation introduced in 4.6 we get that the quadruple

$$(r_{\Omega^x(D)}, r'_{\Omega^x(D)}, s_{\Omega^x(D)}, s'_{\Omega^x(D)}),$$

with

$$\begin{aligned} r_{\Omega^x(D)} &:= (D \otimes p_D^x) \circ t_{D,D} \circ (i_D^x \otimes D), & r'_{\Omega^x(D)} &:= (p_D^x \otimes D) \circ t'_{D,D} \circ (D \otimes i_D^x), \\ s_{\Omega^x(D)} &:= (p_D^x \otimes D) \circ t_{D,D} \circ (D \otimes i_D^x), & s'_{\Omega^x(D)} &:= (D \otimes p_D^x) \circ t'_{D,D} \circ (i_D^x \otimes D), \end{aligned}$$

is a $(\Omega^x(D), D)$ -WO. Indeed, the equalities contained in (c1) and (c2) of Definition 1.9 are a consequence of (162) and (163) as well as the properties of $t_{D,D}$ and $t'_{D,D}$. The proof for the identities of (c3) follows by (17)-(20) and (162), (163). The eight equalities of (c4) follow from (162), (163) and (b3) of Definition 1.5. Finally, (c5) is a consequence of [[2], Proposition 2.12].

For $x = a$ we have that the quadruple $(r_{\Omega^a(D)}, r'_{\Omega^a(D)}, s_{\Omega^a(D)}, s'_{\Omega^a(D)})$ is compatible with the module-comodule structure induced by the action $\varphi_{\Omega^a(D)}$ and the coaction $\rho_{\Omega^a(D)}$. To prove this assertion, by Definition 2.2, for the action $\varphi_{\Omega^a(D)}$ we must show the equalities

$$r_{\Omega^a(D)} \circ (\varphi_{\Omega^a(D)} \otimes D) = (D \otimes \varphi_{\Omega^a(D)}) \circ (t_{D,D} \otimes \Omega^a(D)) \circ (D \otimes r_{\Omega^a(D)}), \quad (165)$$

$$r'_{\Omega^a(D)} \circ (D \otimes \varphi_{\Omega^a(D)}) = (\varphi_{\Omega^a(D)} \otimes D) \circ (D \otimes r'_{\Omega^a(D)}) \circ (t'_{D,D} \otimes \Omega^a(D)) \quad (166)$$

and the analogous equalities taking $t_{D,D}$, $t'_{D,D}$, $s_{\Omega^a(D)}$ and $s'_{\Omega^a(D)}$ instead of $t'_{D,D}$, $t_{D,D}$, $r'_{\Omega^a(D)}$ and $r_{\Omega^a(D)}$ respectively. The proofs for the four equalities are similar and then we only write one of them, for example (166):

$$\begin{aligned} & r'_{\Omega^a(D)} \circ (D \otimes \varphi_{\Omega^a(D)}) \\ &= (p_D^a \otimes D) \circ t'_{D,D} \circ (D \otimes \varphi_D) \circ (D \otimes D \otimes i_D^a) \\ &= ((p_D^a \circ \varphi_D) \otimes D) \circ (D \otimes t'_{D,D}) \circ (t'_{D,D} \otimes i_D^a) \\ &= (\varphi_{\Omega^a(D)} \otimes D) \circ (D \otimes r'_{\Omega^a(D)}) \circ (t'_{D,D} \otimes \Omega^a(D)). \end{aligned}$$

In the last computations, the first and the third equalities follow by (163) and the idempotent character of ω_D^a . The second one is a consequence of (150).

For the coaction $\rho_{\Omega^a(D)} = (D \otimes p_D^a) \circ \delta_D \circ i_D^a$ the proofs for

$$(D \otimes \rho_{\Omega^a(D)}) \circ r_{\Omega^a(D)} = (t_{D,D} \otimes \Omega^a(D)) \circ (D \otimes r_{\Omega^a(D)}) \circ (\rho_{\Omega^a(D)} \otimes D), \quad (167)$$

$$(\rho_{\Omega^a(D)} \otimes D) \circ r'_{\Omega^a(D)} = (D \otimes r'_{\Omega^a(D)}) \circ (t'_{D,D} \otimes \Omega^a(D)) \circ (D \otimes \rho_{\Omega^a(D)}), \quad (168)$$

and the analogous equalities taking $t_{D,D}$, $t'_{D,D}$, $s_{\Omega^a(D)}$ and $s'_{\Omega^a(D)}$ instead of $t'_{D,D}$, $t_{D,D}$, $r'_{\Omega^a(D)}$ and $r_{\Omega^a(D)}$ respectively, are a direct consequence of (162), (163), (b3-3), (b3-4), (23) and (24).

Finally, the triple $(\Omega^a(D), \varphi_{\Omega^a(D)}, \rho_{\Omega^a(D)})$ is a left-left Yetter-Drinfeld module over D because it satisfies (yd3). Indeed:

$$(\mu_D \otimes \Omega^a(D)) \circ (D \otimes r_{\Omega^a(D)}) \circ (((\mu_D \otimes \varphi_{\Omega^a(D)}) \circ (D \otimes t_{D,D} \otimes D)) \circ (\delta_D \otimes \rho_{\Omega^a(D)})) \otimes \lambda_D$$

$$\begin{aligned}
& \circ (D \otimes s_{\Omega^a(D)}) \circ (\delta_D \otimes \Omega^a(D)) \\
& = (\mu_D \otimes p_D^a) \circ (D \otimes t_{D,D}) \circ (((\mu_D \otimes \varphi_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)) \otimes \lambda_D) \circ (D \otimes t_{D,D}) \\
& \quad \circ (\delta_D \otimes i_D^a) \\
& = (D \otimes p_D^a) \circ \delta_D \circ \varphi_D \circ (D \otimes i_D^a) \\
& = \rho_{\Omega^a(D)} \circ \varphi_{\Omega^a(D)},
\end{aligned}$$

where the first equality follows from (157) and (162), the second one by (154) of and the last one by (158).

The proof for the second assertion is similar and we leave the details to the reader. \square

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